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THE NEIGHBOR GRAPH OF BINARY SELF-ORTHOGONAL CODES

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Abstract. We define the neighbor graph of binary self-orthogonal codes, where two codes are connected by an edge if they can be reached by the neighbor construction. We show that this graph consists of two connected, regular subgraphs consisting of self-orthogonal codes that contain the all-one vector $\mathbf{1}$ and self-orthogonal codes that do not contain the all-one vector $\mathbf{1}$. We count the number of vertices and edges in each and give the degree of the vertices.

1. Introduction.

Two of the most important and widely studied classes of codes are self-dual codes and linear complementary dual codes. That is, codes that are equal to their orthogonal and codes that have a minimal intersection with their orthogonal. This inspired the question raised first formally in [5] which is to classify codes that have a k dimensional intersection with their orthogonal. Since the intersection, called the hull, is a self-orthogonal code, it becomes vital to understand all self-orthogonal codes, their relationship to other self-orthogonal codes, and their relationship to codes for which they are the hull.

The principal technique we use is to study these codes using simple graphs. Specifically, we use the distance in an associated graph to define the distance between self-orthogonal codes.

The neighbor construction was first given for self-dual codes and was done very early in the study of self-dual codes. Specifically, neighbor codes in that setting were codes that shared a subcode of co-dimension 1. The construction was used to find new self-dual codes from known self-dual codes. Moreover, it could be used to study various properties of the codes by examining these properties in neighboring codes. A summary of the techniques and uses of the neighbor in the classical case can be found in [7].

In this paper, we shall study self-orthogonal codes and their relationship to each other. We generalize results given for self-dual codes in [6].

In Section 2, we give the necessary notations and definitions for codes and graphs. We define the hull of a code, the shadow of a self-orthogonal code, and give foundational results about both of these codes. In Section 3, we count the number of self-orthogonal codes splitting the number into self-orthogonal codes that contain the all-one vector $\mathbf{1}$ and those that do not contain the all-one vector $\mathbf{1}$, and we

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count the number of codes which have a given self-orthogonal code as their hull. In Section 4, we define the neighbor graph and determine the parameters of this graph. We show that the graph splits into two connected graphs, the first corresponds to codes containing the all-one vector **1** and the second corresponds to codes that do not contain the all-one vector **1**. We show that they are both regular connected subgraphs.

2. **Definitions and notations.** In this section, we set the standard definitions and notations for codes and for simple graphs. Throughout this paper, we restrict ourselves to binary codes.

2.1. **Binary codes.** A binary code of length n is a subset of $\mathbb{F}_2^n$. When the code is a vector subspace of $\mathbb{F}_2^n$, we say that the code is linear. The Hamming weight of a vector is the number of non-zero coordinates in a vector and the minimum weight of a code is the smallest non-zero weight of any vector in the code. We denote the Hamming weight of a vector by $wt_H(\mathbf{v})$. A code in $\mathbb{F}_2^n$, with dimension k, and minimum distance d, is denoted as an $[n, k, d]$ code. We say that two codes C and C' are equivalent, written $C \sim C'$, if and only if there exists a permutation σ acting on the coordinates of C such that $\sigma(C) = C'$.

We attach to the ambient space $\mathbb{F}_2^n$ the standard inner-product, namely $[\mathbf{v}, \mathbf{w}] = \sum v_i w_i$. The dual of the code C, denoted by $C^\perp$, is defined as $C^\perp = \{\mathbf{v} \mid [\mathbf{v}, \mathbf{w}] = 0, \forall \mathbf{w} \in C\}$. The code $C^\perp$ is linear even if C is not, and if C has dimension k then $C^\perp$ has dimension $n - k$.

In this paper, the codes that we are interested in are self-orthogonal codes which we now define.

Definition 2.1. A code $C \subseteq \mathbb{F}_2^n$ is a self-orthogonal code of length n if $C \subseteq C^\perp$. A self-orthogonal code C with the property that there is no self-orthogonal code D with $C \subsetneq D$ is said to be maximal self-orthogonal. A self-dual code satisfies $C = C^\perp$.

We note that a maximal binary self-orthogonal code of odd length n has dimension $\frac{n-1}{2}$ and a maximal binary self-orthogonal code of even length n has dimension $\frac{n}{2}$. In the case when n is even, a maximal self-orthogonal code is, in fact, self-dual, but in the odd case it is not self-dual.

Definition 2.2. For a linear code C, define the Hull of a code C as $Hull(C) = C \cap C^\perp$.

It is immediate that $Hull(C)$ is a linear code and that $Hull(C) \subseteq Hull(C)^\perp$, that is $Hull(C)$ is a self-orthogonal code. If C is self-orthogonal then $Hull(C) = C$. If $C = C^\perp$, that is C is self-dual, we have that $Hull(C) = C$. If $Hull(C) = \{\mathbf{0}\}$ then the code C is said to be Linear Complementary Dual (LCD) and has the property that $C \oplus C^\perp = \mathbb{F}_2^n$. The hull first appeared in [1] where the hull was used to study codes arising from the incidence matrices of finite affine and projective planes. It was also used to study codes from nets in [4] and later was used in general for codes of designs in [2].

If $Hull(C) = C$ and $dim(C) = \frac{n}{2}$ then the code C is self-dual. If $Hull(C) = \{\mathbf{0}\}$, then the code is said to be a Linear Complementary Dual (LCD) code.

Definition 2.3. A code of length n, size M, and minimum Hamming distance d is said to be optimal if it has the largest d of any other codes with length n and size M.

In general, the main question of coding theory is finding optimal codes for a given set of parameters.

The motivating question for this work is to aid in the classification of codes whose hull has dimension k. Fundamentally, we seek to answer the following.

Question 2.1. *What are the parameters of the optimal code C where $dim(Hull(C)) = k$, for a given integer k?*

A great deal of work has been done when $k = \frac{n}{2}$ and more recently when $k = 0$. However, the general question is largely wide open. Specifically, self-dual codes and LCD codes are widely studied classes of codes, but a classification of codes whose hull has dimension $k \neq 0, \frac{n}{2}$ is largely untouched at his point.

2.2. **Subcodes and shadows.** In this section, we recall the notion of a shadow code given in [3] for self-orthogonal codes.

A codeword is said to be doubly-even if its weight is congruent to $0 \pmod 4$. A self-orthogonal code with the property that all weights are doubly-even is said to be a doubly-even code. The weight enumerator of a code is given by:

$$W_C(x, y) = \sum A_i x^{n-i} y^i$$

where A_i is the number of vectors of weight i.

Definition 2.4. If C is a self-orthogonal code, define the code C_0 to be $C_0 = \{\mathbf{v} \in C \mid wt(\mathbf{v}) \equiv 0 \pmod 4\}$.

Theorem 2.5. *Let C be a self-orthogonal code of dimension k. Then C_0 is a linear subcode of C with co-dimension 1, that is a subcode of dimension $k - 1$.*

Proof. We have $wt_H(\mathbf{v} + \mathbf{w}) = wt_H(\mathbf{v}) + wt_H(\mathbf{w}) - 2|\mathbf{v} \wedge \mathbf{w}|$, where $|\mathbf{v} \wedge \mathbf{w}|$ is the number of coordinates where $v_i = w_i = 1$. If $[\mathbf{v}, \mathbf{w}] = 0$, this implies that $|\mathbf{v} \wedge \mathbf{w}| \equiv 0 \pmod 2$.

Let $\mathbf{v}$ and $\mathbf{w}$ be elements of C_0, that is they are two doubly-even vectors that are orthogonal. Then $wt_H(\mathbf{v})$ and $wt_H(\mathbf{w})$ are both $0 \pmod 4$. Since $|\mathbf{v} \wedge \mathbf{w}| \equiv 0 \pmod 2$, we have $2|\mathbf{v} \wedge \mathbf{w}| \equiv 0 \pmod 4$. Therefore, $wt_H(\mathbf{v} + \mathbf{w}) \equiv 0 \pmod 4$ and $\mathbf{v}, \mathbf{w} \in C_0$. This gives that C_0 is linear.

Next, if $\mathbf{u}$ is any singly-even vector in C, we have $\langle C_0, \mathbf{u} \rangle = C$. This gives that C_0 has co-dimension 1 in C. □

We note that it is vital that the code C be self-orthogonal for the previous theorem to hold. For example, consider the code E_5 consisting of all even weight vectors in $\mathbb{F}_2^5$. Then (11110) and (01111) are both doubly-even vectors in that code but their sum (10001) is not a doubly-even vector. Moreover, there are 6 doubly-even vectors in E_5 and 6 is not a power of 2. As such, while you can generate a linear code containing all even vectors in the ambient space, it is not necessarily true for doubly-even vectors. Namely, the code containing all doubly-even vectors is not linear in general.

Definition 2.6. The shadow of a self-orthogonal code C is $S = C_0^{\perp} \setminus C^{\perp}$.

The following was proven in [3].

Theorem 2.7. *Let C be a self-orthogonal code with weight enumerator $W_C(x, y)$ and let i be the complex root of -1.*

- *The weight enumerator of C_0 is $W_{C_0}(x, y) = \frac{1}{2}(W_C(x, y) + W_C(x, iy))$.*

- *The shadow S is a non-linear code with $|S| = 2^{n-k}$.*
- *The weight enumerator of the shadow is*

$$\begin{aligned} W_S(x,y) =& \frac{1}{2^{n-k}} W_{C_0}(x+y, x-y) \\ =& W_C\left(\frac{(1+i)x+(1-i)y}{2}, \frac{(1-i)x+(1+i)y}{2}\right). \end{aligned}$$

With these results in mind, we can make the following definition.

Definition 2.8. Let C be a binary linear code. Then $Hull(C)_0$ is the subcode of doubly-even vectors in $Hull(C)$. The non-linear code $Shad(C) = Hull(C)_0^\perp \setminus Hull(C)^\perp$.

This means that given any binary code there is a self-orthogonal code, and doubly-even code, and a non-linear code attached to it.

Theorem 2.9. *Let C and C' be two binary codes that are equivalent. Then we have the following.*

- *The codes $Hull(C)$ and $Hull(C')$ are equivalent.*
- *The codes $Hull(C)_0$ and $Hull(C')_0$ are equivalent.*
- *The codes $Shad(C)$ and $Shad(C')$ are equivalent.*

Proof. If $C \sim C'$ then it is immediate that $C^\perp \sim (C')^\perp$ since if $\sigma(C) = C'$ then $\sigma(C^\perp) = (C')^\perp$. This gives that $Hull(C)$ and $Hull(C')$ and the other two statements follow immediately from that result. □

For LCD codes C, we have $Hull(C)_0 = Hull(C) = \{\mathbf{0}\}$ and for self-dual codes C we have $Hull(C)_0 = C_0$.

Corollary 2.10. *Let C and C' be two codes. If $W_{Hull(C)}(x,y) \neq W_{Hull(C')}(x,y)$, then C and C' are not equivalent.*

Proof. Follows from Theorem 2.9. □

2.3. **Simple graphs.** There are various different notions of a graph. In this setting, we shall describe what are usually known as simple graphs. That is, there are no multiple edges, no edges from a vertex to itself, and the edges are not directed. Specifically, we have the following definition.

Definition 2.11. A graph $G = (V, E)$ where V is the set of vertices and E is the set of edges of the form $\{a, b\} \in E$, where $a \neq b$.

We say that a and b are connected as edges if and only if $\{a, b\} \in E$. We note also that $\{a, b\}$ is a set and not an ordered pair, meaning it is not a directed graph and also that E is a set and not a multi-set so there are no multiple edges.

The degree of a vertex is the number of edges adjacent to it and a graph is said to be regular if the degree of every vertex is the same. A path in a graph G is a set of vertices $v_1, v_2, \ldots, v_r$ where $\{v_i, v_{i+1}\} \in E$. In this case, we say that it is a path of length $r-1$. We say that a graph is connected if, for all vertices $v, w \in V$, there is a path from v to w. If a graph G is connected and regular of degree d, then $|E| = \frac{|V|d}{2}$.

3. **Counting self-orthogonal codes.** Let $\mathbf{1}$ denote the all-one vector of length n. We note that a vector is self-orthogonal if and only if $\mathbf{v}$ has $[\mathbf{v},\mathbf{v}]=0$ if and only if $[\mathbf{1},\mathbf{v}]=0$. Note also that $[\mathbf{1},\mathbf{1}]=0$ if and only if n is even.

Lemma 3.1. *Let n be odd. The number of self-orthogonal codes of dimension k, with $k<\frac{n}{2}$ is*

$$\begin{aligned}\mathcal{O}_{n,k} &= \frac{(2^{n-1}-1)(2^{n-2}-2)(2^{n-3}-2^2)\cdots(2^{n-k}-2^{k-1})}{(2^k-1)(2^k-2)(2^k-2^2)\cdots(2^k-2^{k-1})}\\ &= \frac{\prod_{i=1}^{k}(2^{n-i}-2^{i-1})}{\prod_{i=1}^{k}(2^k-2^{i-1})}.\end{aligned}$$

Proof. All binary self-orthogonal vectors are in $\langle\mathbf{1}\rangle^\perp$ which has cardinality 2^{n-1}. The number of ways of choosing a non-zero vector from this set is $2^{n-1}-1$. Having chosen vector $\mathbf{v}$ we next have to choose a vector from $\langle\mathbf{v},\mathbf{1}\rangle^\perp$ that is not in $\langle\mathbf{v}\rangle$. There are $(2^{n-2}-2)$ ways of doing that. Having chosen $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_s$ we must choose the next vector from $\langle\mathbf{1},\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_s\rangle^\perp$ that is not in $\langle\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_s\rangle$. There are $(2^{n-1}-1)(2^{n-2}-2)\ldots(2^{n-1-s}-2^s)$ ways of doing this. This gives the numerator. The denominator counts the number of ways of finding a basis in a k dimensional space. □

Notice in the above formula, that is $n=2k-1$, we have that $2^{n-k}-2^{k-1}=2^{k-1}-2^{k-1}=0$. That is, there are no self-orthogonal codes with dimension larger than $\lfloor\frac{n}{2}\rfloor$ when n is odd as is well known.

We split the counting of self-orthogonal codes into two parts consisting of codes containing $\mathbf{1}$ and codes that do not contain $\mathbf{1}$. This split will be essential later as both counts are important for future results.

Lemma 3.2. *Let n be even. The number of self-orthogonal codes of dimension k, with $k<\frac{n}{2}$, that do not contain $\mathbf{1}$ is*

$$\begin{aligned}\mathcal{H}_{n,k} &= \frac{(2^{n-1}-2)(2^{n-2}-2^2)(2^{n-3}-2^3)\cdots(2^{n-k}-2^{k})}{(2^k-1)(2^k-2)(2^k-2^2)\cdots(2^k-2^{k-1})}\\ &= \frac{\prod_{i=1}^{k}(2^{n-i}-2^{i})}{\prod_{i=1}^{k}(2^k-2^{i-1})}.\end{aligned}$$

Proof. All binary self-orthogonal vectors are in $\langle\mathbf{1}\rangle^\perp$ which has cardinality 2^{n-1}. The number of ways of choosing a non-zero vector which is not the all-one vector from this set is $2^{n-1}-2$. Having chosen vector $\mathbf{v}$ we next have to choose a vector from $\langle\mathbf{v},\mathbf{1}\rangle^\perp$ that is not in $\langle\mathbf{v},\mathbf{1}\rangle$. This is, because we cannot choose the all-one vector. There are $(2^{n-2}-2^2)$ ways of doing that. Having chosen $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_s$ we must choose the next vector from $\langle\mathbf{1},\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_s\rangle^\perp$ that is not in $\langle\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_s\rangle$. There are $(2^{n-1}-2)(2^{n-2}-2^2)\cdots(2^{n-s+1}-2^{s+1})$ ways of doing this. This gives the numerator. The denominator counts the number of ways of finding a basis in a k dimensional space. □

We note that $\mathcal{H}_{n,0}=1$ which is the code generated by the all-zero vector.

Lemma 3.3. *Let n be even. The number of self-orthogonal codes of dimension k, with $k\le\frac{n}{2}$, that do contain $\mathbf{1}$ is $\frac{\mathcal{H}_{n,k-1}}{2^{k-1}}$.*

Proof. A self-orthogonal code that contains the all-one vector is a self-orthogonal code D of dimension $k-1$ adjoined with the all-one vector. That is $C=\langle D,\mathbf{1}\rangle$.

For each vector $\mathbf{v}$ in the dimension $k-1$ code C, we get the same code by adjoining $\mathbf{1}+\mathbf{v}$. Therefore, 2^{k-1} vectors given the same code $\langle C, \mathbf{1}\rangle$. This is why we divide by 2^{k-1}. □

Theorem 3.4. *Let n be even. The number of self-orthogonal codes of dimension k is*

$$\mathcal{H}_{n,k} + \frac{\mathcal{H}_{n,k-1}}{2^{k-1}}.$$

Proof. We add the number of self-orthogonal codes without the all-one vector together with the self-orthogonal codes that have the all-one vector. □

Note that for self-dual codes, that is self-orthogonal codes with $k = \frac{n}{2}$, we have $\mathcal{H}_{n,k} = 0$, since $2^{n-k} - 2^k = 0$ as $n - k = k$ in this case. This coincides with the fact that all self-dual codes must contain the all-one vector.

Theorem 3.5. *If n is odd, the number of self-orthogonal codes of dimension k is:*

$$\mathcal{O}_{n,k} = \frac{(2^{n-1}-1)(2^{n-3}-1)\cdots(2^{n-(2k-1)}-1)}{(2^k-1)(2^{k-1}-1)\cdots(2-1)} = \prod_{i=0}^{k-1}\frac{(2^{n-1-2i}-1)}{(2^{k-i}-1)}. \quad (1)$$

If n is even, the number of self-orthogonal codes of dimension k that do not contain the vector $\mathbf{1}$ is:

$$\mathcal{H}_{n,k} = \frac{2^k(2^{n-2}-1)(2^{n-4}-1)\cdots(2^{n-2k}-1)}{(2^k-1)(2^{k-1}-1)\cdots(2-1)} = 2^k\prod_{i=0}^{k-1}\frac{(2^{n-2(i+1)}-1)}{(2^{k-i}-1)}. \quad (2)$$

If n is even, the number of self-orthogonal codes of dimension k is:

$$\mathcal{E}_{n,k} = \left(2^k\prod_{i=0}^{k-1}\frac{(2^{n-2(i+1)}-1)}{(2^{k-i}-1)}\right) + \left(\prod_{i=0}^{k-2}\frac{(2^{n-2(i+1)}-1)}{(2^{k-i-1}-1)}\right). \quad (3)$$

Proof. For $\mathcal{O}_{n,k}$, take the value given in Lemma 3.1 and factor the powers of 2 out of the top and the bottom to get the result.

For $\mathcal{H}_{n,k}$, take the value given in Lemma 3.2 and factor the powers of 2 out of the top and the bottom which leaves 2^k in the numerator to get the result.

For $\mathcal{E}_{n,k}$, use the result in Equation 2 and compute $\mathcal{H}_{n,k} + \frac{\mathcal{H}_{n,k-1}}{2^{k-1}}$. □

Table 1 gives the number of binary self-orthogonal codes for some lengths. Note that $\mathcal{H}_{n,k}$ is the number of self-orthogonal codes of length n and dimension k without the all-one vector, $\mathcal{H}_{n,k-1}/2^{k-1}$ is the number of self-orthogonal codes of length n and dimension k with the all-one vector and $\mathcal{E}_{n,k}$ is the total number of self-orthogonal codes of length n and dimension k, where n is even. Note also that $\mathcal{O}_{n,k}$ is the total number of self-orthogonal codes of odd length. We have that $\mathcal{H}_{n,k-1}/2^{k-1}$ gives the number of self-dual codes of even length n, when $k = n/2$. Moreover, it can be seen from the table and from the formulas given by (1) and (3) that the number of binary self-dual codes of length n and dimension k is equal to the number of binary maximal self-orthogonal codes of length $n-1$ and dimension $k-1$, where $k = \frac{n}{2}$. These numbers are bolded in the table and are given as follows: $1, 3, 15, 135, 2295, 75735, \ldots$ This is a special number sequence given in OEIS by reference number $A028362$, in [9]. This sequence also gives the number of totally isotropic spaces of index n in symplectic geometry of even dimension n.

TABLE 1. The number of binary self-orthogonal codes length n and dimension k

n	**k**	$\mathcal{O}_{n,k}$	**n**	**k**	$H_{n,k}$	$\frac{H_{n,k-1}}{2^{k-1}}$	$\mathcal{E}_{n,k}$
1	0	**1**	2	0	1	1	2
3	0	1	2	1		**1**	
3	1	**3**	4	0	1	1	2
5	0	1	4	1	6	1	7
5	1	15	4	2		**3**	
5	2	**15**	6	0	1	1	2
7	0	1	6	1	30	1	31
7	1	63	6	2	60	15	75
7	2	315	6	3		**15**	
7	3	**135**	8	0	1	1	2
9	0	1	8	1	126	1	127
9	1	255	8	2	1260	63	1323
9	2	5355	8	3	1080	315	1395
9	3	11475	8	4		**135**	
9	4	**2295**	10	0	1	1	2
11	0	1	10	1	510	1	511
11	1	1023	10	2	21420	255	21675
11	2	86955	10	3	91800	5355	97155
11	3	782595	10	4	36720	11475	48195
11	4	782595	10	5		**2295**	
11	5	**75735**	12	0	1	1	2
13	0	1	12	1	2046	1	2047
13	1	4095	12	2	347820	1023	348843
13	2	1396395	12	3	6260760	86955	63477153
13	3	50868675	12	4	12521520	782595	13304115
13	4	213648435	12	5	2423520	782595	3206115
13	5	103378275	12	6		**75735**	
13	6	4922775	12				

3.1. **Codes and hulls.** Let $\mathcal{T}_{n,k}$ be $\mathcal{O}_{n,k}$ if n is odd and $\mathcal{E}_{n,k}$ if n is even. That is, $\mathcal{T}_{n,k}$ is the number of self-orthogonal codes with dimension k and length n.

Sendrier proved the following result in [8].

Lemma 3.6. *Let $j \leq \frac{n}{2}$ and let $k \leq j$. The number of binary linear codes of length n and dimension j where the dimension of the hull is k is*

$$\sum_{i=k}^{j} \begin{bmatrix} n-2i \\ j-i \end{bmatrix}_2 \begin{bmatrix} i \\ k \end{bmatrix}_2 (-1)^{i-k} 2^{\binom{i-k}{2}} \mathcal{T}_{n,i}.$$

Theorem 3.7. *Let C be a self-orthogonal code of dimension k.*

1. *Let n be odd. Then the number of binary codes D with $Hull(D) = C$ is:*

$$\left(\frac{2}{\mathcal{O}_{n,k}}\right) \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{i=k}^{j} \begin{bmatrix} n-2i \\ j-i \end{bmatrix}_2 \begin{bmatrix} i \\ k \end{bmatrix}_2 (-1)^{i-k} 2^{\binom{i-k}{2}} \mathcal{O}_{n,i} \right). \tag{4}$$

2. *Let n be even. Then the number of binary codes D with $Hull(D) = C$ is:*

$$\left(\frac{1}{\mathcal{E}_{n,k}}\right) \left(2 \sum_{j=k}^{\frac{n}{2}-1} \left(\sum_{i=k}^{j} \begin{bmatrix} n-2i \\ j-i \end{bmatrix}_2 \begin{bmatrix} i \\ k \end{bmatrix}_2 (-1)^{i-k} 2^{\binom{i-k}{2}} \mathcal{E}_{n,i} \right) \right.$$
$$\left. + \sum_{i=k}^{\frac{n}{2}} \begin{bmatrix} n-2i \\ \frac{n}{2}-i \end{bmatrix}_2 \begin{bmatrix} i \\ k \end{bmatrix}_2 (-1)^{i-k} 2^{\binom{i-k}{2}} \mathcal{E}_{n,i} \right).$$

Proof. For n even or odd we have the following. Since

$$\sum_{i=k}^{j} \begin{bmatrix} n-2i \\ j-i \end{bmatrix}_2 \begin{bmatrix} i \\ k \end{bmatrix}_2 (-1)^{i-k} 2^{\binom{i-k}{2}} \mathcal{T}_{n,i}$$

gives the number of codes of dimension $j \leq \lfloor \frac{n}{2} \rfloor$ that have a hull with dimension $k = dim(C)$, the number of codes D with a hull of dimension $\lfloor \frac{n}{2} \rfloor$ is

$$\sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{i=k}^{j} \begin{bmatrix} n-2i \\ j-i \end{bmatrix}_2 \begin{bmatrix} i \\ k \end{bmatrix}_2 (-1)^{i-k} 2^{\binom{i-k}{2}} \mathcal{T}_{n,i} \right).$$

If $C = Hull(D)$ then $Hull(D^{\perp}) = C$ as well. Either D or $D^{\perp}$ has dimension less than or equal to $\lfloor \frac{n}{2} \rfloor$. This gives that the number of codes with a hull of dimension k is

$$2 \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{i=k}^{j} \begin{bmatrix} n-2i \\ j-i \end{bmatrix}_2 \begin{bmatrix} i \\ k \end{bmatrix}_2 (-1)^{i-k} 2^{\binom{i-k}{2}} \mathcal{T}_{n,i} \right).$$

Then there are $\mathcal{T}_{n,k}$ self-orthogonal code of dimension k, each of these has the same number of codes D for which it is the hull and this gives the $\mathcal{T}_{n,k}$ in the denominator.

For the case when n is even, if $dim(C) = \frac{n}{2}$, then its orthogonal must also have dimension $\frac{n}{2}$ so we do not need to multiply by 2. Then we have the result. □

4. **Neighbor graph.** In this section, we shall investigate the neighbor of a self-orthogonal code. We begin with the definition of a neighbor.

Definition 4.1. Let C be a self-orthogonal code of dimension k and let $\mathbf{v}$ be a self-orthogonal vector not in $C^{\perp}$. Then $N(C, \mathbf{v}) = \langle \{\mathbf{w} \in C \mid [\mathbf{w}, \mathbf{v}] = 0\}, \mathbf{v} \rangle$.

The significance of this definition is that given a self-orthogonal code C, we can produce another self-orthogonal code given any self-orthogonal vector not in the code $C^{\perp}$. We describe this process in the next theorem.

Theorem 4.2. *Let C be a self-orthogonal code of dimension k and let $\mathbf{v}$ be a self-orthogonal vector not in $C^{\perp}$. Then $N(C, \mathbf{v})$ is a self-orthogonal code of dimension k.*

Proof. If C is a linear self-orthogonal code then $\{\mathbf{w} \in C \mid [\mathbf{w}, \mathbf{v}] = 0\}$ is a linear code of co-dimension 1 in C. Then, taking the code generated by this linear code and a vector $\mathbf{v}$ that is not in the code gives a linear code with dimension k. Therefore, $N(C, \mathbf{v})$ is a linear code with dimension k.

We have that $[\mathbf{v}, \mathbf{v}] = 0$ and $[\mathbf{v}, \mathbf{w}] = 0$ for all $\mathbf{w} \in C_0$. This gives that $N(C, \mathbf{v})$ is a self-orthogonal code. □

It is vital that $\mathbf{v}$ not be in $C^{\perp}$, since if it were then $\{\mathbf{w} \in C \mid [\mathbf{w}, \mathbf{v}] = 0\}$ would be the code C and then $N(C, \mathbf{v})$ would be dimension $k + 1$ and not k.

The code we have constructed $N(C, \mathbf{v})$ is called a neighbor of the code C. We note that if $\mathbf{v} \notin C$ then $N(C, \mathbf{v}) \neq C$.

Each code has numerous neighbors and simply because $\mathbf{v}_1 \neq \mathbf{v}_2$ does not imply that $N(C, \mathbf{v}_1) \neq N(C, \mathbf{v}_2)$. We shall describe the situation in the following lemma which generalizes a lemma in [6].

Lemma 4.3. *Let C be a binary self-orthogonal code and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be self-orthogonal vectors that are not in the code C. We have $N(C, \mathbf{v}_1) = N(C, \mathbf{v}_2)$ if and only if there exists a vector $\mathbf{w} \in C$ with the properties that $[\mathbf{w}, \mathbf{v}_1] = 0$ and $\mathbf{v}_2 = \mathbf{w} + \mathbf{v}_1$.*

Proof. Assume there exists a vector $\mathbf{w} \in C$ with $[\mathbf{w}, \mathbf{v}_1] = 0$ and $\mathbf{v}_2 = \mathbf{w} + \mathbf{v}_1$. Let C_0 be the subcode of C that is $\{\mathbf{w} \in C \mid [\mathbf{w}, \mathbf{v}_1] = 0\}$. If $\mathbf{u} \in C_0$ we have

$[\mathbf{u}, \mathbf{v}_2] = [\mathbf{u}, \mathbf{w} + \mathbf{v}_1] = 0 + 0 = 0$. Thus, $\{\mathbf{w} \in C \mid [\mathbf{w}, \mathbf{v}_2] = 0\} = C_0$. Then $C_0 + \mathbf{v}_2 = C_0 + \mathbf{w} + \mathbf{w}_1 = (C_0 + \mathbf{w}) + \mathbf{v}_1 = C_0 + \mathbf{v}_1$ giving that $N(C, \mathbf{v}_1) = N(C, \mathbf{v}_2)$.

Assume $N(C, \mathbf{v}_1) = N(C, \mathbf{v}_2)$, then $\mathbf{v}_2 \in C_0 + \mathbf{v}_1$ which gives that there exists a vector $\mathbf{w} \in C$ with $[\mathbf{w}, \mathbf{v}_1] = 0$ and $\mathbf{v}_2 = \mathbf{w} + \mathbf{v}_1$. □

The following is the motivating definition for the work.

Definition 4.4. Let n and k be positive natural numbers with $k \leq \frac{n}{2}$. Let $\Gamma_{n,k} = (V, E)$, where the set of vertices V is the set of all binary self-orthogonal codes of length n and dimension k, and two vertices are connected by an edge in E if and only if they are neighbors.

We want to justify that the graph is a simple graph and not a directed graph. We show this in the next theorem.

Theorem 4.5. *Let C be a binary self-orthogonal code of dimension k. If $C' = N(C, \mathbf{v})$ for some vector $\mathbf{v}$, then $C = N(C', \mathbf{w})$ for some vector $\mathbf{w}$.*

Proof. Let C be a binary self-orthogonal code and $\mathbf{v}$ a self-orthogonal vector, and let $C_0 = \{\mathbf{c} \in C \mid [\mathbf{c}, \mathbf{v}] = 0\}$. This gives that there is a vector $\mathbf{w}$ with $\langle C_0, \mathbf{w}\rangle = C$. We note that the vector $\mathbf{w}$ is orthogonal to every vector in C_0.

Next, consider the code $N(C', \mathbf{w})$. The code $\{\mathbf{c} \in C \mid [\mathbf{c}, \mathbf{w}] = 0\}$ is necessarily C_0 since this code must be co-dimension 1 in C' and $\mathbf{w}$ is orthogonal to every vector in C_0. Then $N(C', \mathbf{w}) = \langle C_0, \mathbf{w}\rangle = C$ and we have the result. □

Unlike the self-dual code case, simply because two codes share a subcode of co-dimension 1, they are not necessarily neighbors. For example consider the code of dimension 1, $C = \langle(111111)\rangle$ and the code $C' = \langle(111100)\rangle$. They share a subcode of co-dimension 1, namely $H = \{(000000)\}$. However, $C^{\perp}$ consists of every self-orthogonal vector in $\mathbb{F}_2^6$. Hence there is no self-orthogonal vector to make C and C' neighbors. That is $\{\mathbf{w} \mid \mathbf{w} \in C, [\mathbf{w}, (111100)] = 0\}$ is the code C itself. Therefore, $N(C, (111100))$ is a code of dimension 2 and not dimension 1.

Instead, we have the following result.

Theorem 4.6. *Two self-orthogonal codes C and D of dimension k and length n are connected by an edge in $\Gamma_{n,k}$ if they share a subcode E of dimension $k-1$ where $C = \langle E, \mathbf{v}\rangle$, $D = \langle E, \mathbf{w}\rangle$, and $[\mathbf{v}, \mathbf{w}] \neq 0$.*

Proof. If $C = \langle E, \mathbf{v}\rangle$, $D = \langle E, \mathbf{w}\rangle$, and $\mathbf{v} \notin D^{\perp}$, then $N(D, \mathbf{v}) = C$ since the subcode of D that is orthogonal to $\mathbf{v}$ is E and is dimension $k - 1$. Equivalently, $N(C, \mathbf{w}) = D$.

If $[\mathbf{v}, \mathbf{w}] = 0$, then $\mathbf{v} \in D^{\perp}$ and $N(D, \mathbf{v})$ has dimension $k + 1$. □

Corollary 4.7. *Let C and D be maximal self-orthogonal codes, that is $k = \lfloor \frac{n}{2} \rfloor$. Then if C and D share a subcode of co-dimension 1, then they are connected by an edge in $\Gamma_{n,k}$.*

Proof. Assume $C = \langle E, \mathbf{v}\rangle$ and $D = \langle E, \mathbf{w}\rangle$ and C and D are both maximal self-orthogonal. Then if $[\mathbf{v}, \mathbf{w}] = 0$, this would imply that $\langle C, \mathbf{w}\rangle$ is a self-orthogonal code with dimension $\lfloor \frac{n}{2} \rfloor + 1$ which is impossible since self-orthogonal codes satisfy $k \leq \lfloor \frac{n}{2} \rfloor$. Therefore, $[\mathbf{v}, \mathbf{w}] \neq 0$ in this case and so Theorem 4.6 applies. □

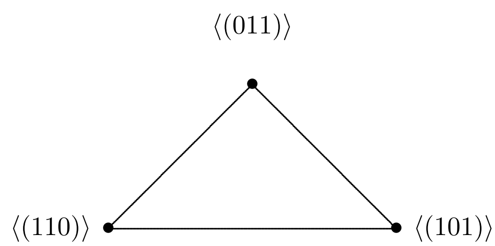
If $k = 0$, then there is only one self-orthogonal code, namely, the code $\{\mathbf{0}\}$. Therefore, $\Gamma_{n,k}$ consists of a single vertex and no edges. If $n = 2$, and $k = 1$, then $\Gamma_{2,1}$ consists of a single vertex corresponding to $\{(00), (11)\}$.

Theorem 4.8. *Let $k = 1$. If n is odd, then $\Gamma_{n,k}$ has $2^{n-1} - 1$ vertices and the graph is regular with degree 2^{n-2}. If n is even, then $\Gamma_{n,k}$ has $2^{n-1} - 1$ vertices and the vertices not of the form $\{\mathbf{0}, \mathbf{1}\}$ has degree 2^{n-2} and the vertex corresponding to $\{\mathbf{0}, \mathbf{1}\}$ is an isolated point.*

Proof. Lemma 3.1 gives the number of vertices in the odd case. Each self-orthogonal code is of the form $\{\mathbf{0}, \mathbf{v}\}$ where $\mathbf{v}$ is a self-orthogonal vector. Then $\{\mathbf{0}, \mathbf{v}\}$ and $\{\mathbf{0}, \mathbf{w}\}$ are connected if and only if $[\mathbf{v}, \mathbf{w}] \neq 0$ by Theorem 4.6. There are $2^{n-2} - 1$ non-zero self-orthogonal vectors in $\langle \mathbf{v}, \mathbf{1} \rangle^{\perp}$, that is self-orthogonal vectors that are orthogonal to $\mathbf{v}$. Therefore, the code $\{\mathbf{0}, \mathbf{v}\}$ is connected to $2^{n-1} - 1 - (2^{n-2} - 1) = 2^{n-2}$ vertices.

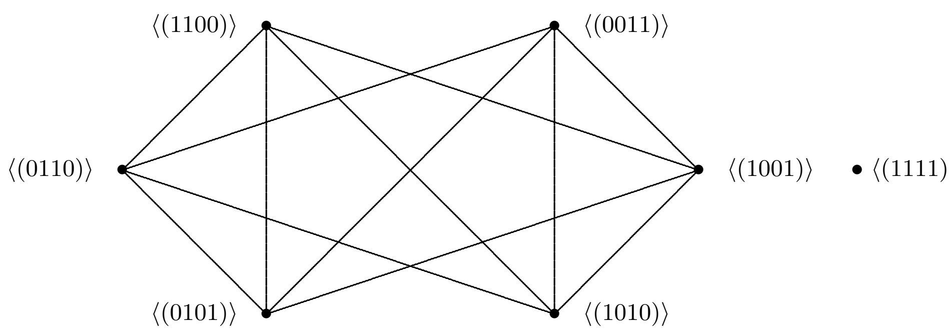
In the even case, Theorem 3.5 gives the number of vertices. For the code $\{\mathbf{0}, \mathbf{1}\}$, every self-orthogonal vector is in its orthogonal so it is not connected to any other vertex. □

Example 1. For $n = 3$, the graph $\Gamma_{3,1}$ is the following:



Here there are 3 vertices and the degree of each vertex is 2.

Example 2. For $n = 4$, the graph $\Gamma_{4,1}$ is the following:



Here there are 7 vertices and the degree of each vertex is 4 in the connected part corresponding to codes that do not contain the all-one vector. The other part of the graph which corresponds to codes that contain the all-one vector consists of a unique isolated vertex. Note that this is not a regular graph but both connected subgraphs are regular.

Definition 4.9. Let n and k be positive natural numbers with $k \leq \frac{n}{2}$, where n is even and $n \geq 4$. Let $\Theta_{n,k} = (V, E)$, where the set of vertices V is the set of all binary self-orthogonal codes that contain the all-one vector of length n and dimension k, and two vertices are connected by an edge in E if and only if they are neighbors.

We see that this graph is only defined for even n since the all-one vector is self-orthogonal if and only if n is even. The graph $\Theta_{n,k}$ has $\frac{\mathcal{H}_{n,k-1}}{2^{k-1}}$ vertices by Lemma 3.3.

Example 3. If $k = 1$, then $\Theta_{n,1}$ has a single vertex corresponding to the code $\{\mathbf{0}, \mathbf{1}\}$.

Example 4. Let $n = 6$ and $k = 2$. Then the number of vertices in $\Theta_{6,2}$ is $\frac{\mathcal{H}_{6,1}}{2^1} = 15$. Consider the code $C = \{000000, 111111, 110000, 001111\}$. Then this code is connected to codes generated by the set $\{111111, \mathbf{v}\}$ where $\mathbf{v}$ is self-orthogonal but not orthogonal to 110000. There are $\binom{4}{1} = 4$ vectors that have weight 2 that begin with 10 and $\binom{4}{1} = 4$ that have weight 2 that begin with 01. There are $\binom{4}{3} = 4$ vectors that have weight 4 and begin with 10 there are $\binom{4}{3} = 4$ vectors that have weight 4 and begin with 01. Each of these codes are counted twice, once for $\mathbf{v}$ and once for $\mathbf{1} + \mathbf{v}$. Therefore, there are 8 codes that this code is connected to by an edge.

The code is not connected to a code generated by the set $\{111111, \mathbf{v}\}$ if $\mathbf{v}$ has weight 2 whose support does not contain the first two coordinates which gives $\binom{4}{2} = 6$ codes or a weight 4 vectors whose support does contain the first two coordinates. This gives $\binom{4}{2} = 6$ codes. Then each code is counted twice so there are 6 codes that are not connected to the code C.

Then there is the code C, the 8 codes that it is connected to, and the 6 codes that it is not connected to, giving $1 + 8 + 6 = 15$ as predicated.

We want to examine the graph theoretic properties of the neighbor graph of self-orthogonal codes.

Theorem 4.10. *The graph $\Gamma_{n,\lfloor \frac{n}{2} \rfloor}$ is connected.*

Proof. Assume C_1, C_2 are maximal self-orthogonal codes. If $\mathbf{v} \in C_2$, then we know that $\mathbf{v} \notin C_1$, then $\mathbf{v} \notin C_1^{\perp}$ since if it were $\langle C_1, \mathbf{v} \rangle$ would be self-orthogonal which is a contradiction. Therefore, if $\mathbf{v} \in C_1$ we have $N(C_1, \mathbf{v}) = C_1$. Let $C_2 = \langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \rangle$. Let $D_1 = N(C_1, \mathbf{v}_1)$ and $D_s = N(D_{s-1}, \mathbf{v}_s)$. Then $C_1, D_1, D_2, \dots, D_k$ is a path from C_1 to C_2 by Corollary 4.7. □

Theorem 4.11. *The maximum distance between any two vertices in $\Gamma_{n,\lfloor \frac{n}{2} \rfloor}$ is k.*

Proof. Let C be a self-orthogonal code and let $D = \langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \rangle$ be another self-orthogonal code. Then

$$D = N(\cdots(N(N(N(C, \mathbf{v}_1), \mathbf{v}_2), \mathbf{v}_3) \cdots), \mathbf{v}_k).$$

Therefore, the maximum distance is k. □

We note that for self-dual codes the maximum distance is $\frac{n}{2} - 1$ since any two self-dual codes must share the one dimensional code generated by the all-one vector. However, it is easy to see that there exist self-orthogonal codes that are not self-dual that do not even share a 1 dimensional subcode. For example, consider the code generated by $(11000\cdots00, 001100\cdots)$ and the code generated by $(00\cdots0011, 00\cdots1100)$. Similar things can be done for any dimension k as long as $k < \frac{n}{2}$.

The set of doubly-even codes of dimension k is a subset of the set of k dimensional self-orthogonal codes. This is because if C is a linear code consisting of doubly-even vectors, then if $\mathbf{v}, \mathbf{w} \in C$, then $\mathbf{v} + \mathbf{w} \in C$ and therefore $wt_H(\mathbf{v} + \mathbf{w}) \equiv 0 \pmod 4$.

Then $wt_H(\mathbf{v}+\mathbf{w}) = \mathrm{wt_H}(\mathbf{v}) + \mathrm{wt_H}(\mathbf{w}) - 2|\mathbf{v} \wedge \mathbf{w}|$ giving that $|\mathbf{v} \wedge \mathbf{w}|$ is even and so $[\mathbf{v}, \mathbf{w}] = 0$.

Theorem 4.12. *Let C be a self-orthogonal code. Then $N(C, \mathbf{v})$ is a doubly-even code if and only if $\mathbf{v} \in C_0^\perp \setminus C^\perp$ and $wt_H(\mathbf{v}) \equiv 0 \pmod 4$.*

Proof. Let C be a Type I code of length n. If $N(C, \mathbf{v})$ is a Type II code, then two things must occur. First $\mathbf{v}$ must be a doubly-even vector since if $\mathbf{v}$ is a singly-even vector then $\mathbf{v} \in N(C, \mathbf{v})$ so $N(C, \mathbf{v})$ must be singly-even. The second is that $C_0 = \{\mathbf{w} \mid [\mathbf{v}, \mathbf{w}] = 0\}$ must consist of only doubly-even vectors. That is, we need $C_0 = D_0$, where D_0 is the subcode of doubly-even vectors.

It is immediate that $C_0 = D_0$ if and only if $\mathbf{v} \in S$, where S is the shadow of the code. □

Definition 4.13. Let $\Delta_n^k = (V, E)$ be the subgraph of $\Gamma_{n,k}$, where V is the set of all doubly-even codes of length n, and two vertices are connected by an edge in E if and only if they are neighbors.

The difficulty in determining all of the parameters of $\Delta_{n,k}$ is that there is no formula for the number of doubly-even codes of dimension k. If we were to follow the techniques used to get the counts in Section 3, we would need to know the following. Let $\mathfrak{D}_n$ be the set of all doubly-even vectors in $\mathbb{F}_2^n$ and let $D_n = |\mathfrak{D}_n|$. The question is, given a doubly-even vector what the size of the set $\mathfrak{D}_n \cap \langle \mathbf{v} \rangle^\perp$ is. However, this is not the same for all vectors $\mathbf{v}$ even for a given n. For example, if $n = 8$ and $\mathbf{v} = \mathbf{1}$ then there are $D_8 = 72$ doubly even vectors in $\mathfrak{D}_n \cap \langle \mathbf{v} \rangle^\perp$ since every doubly-even vector is in $\langle \mathbf{1} \rangle^\perp$ for this n. However, choosing $\mathbf{v} = (11110000)$, not every doubly-even vector is orthogonal to this vector. For example, (10001110) is not. Hence, the size of this intersection is not the same and therefore any attempt to count the number of doubly-even codes of dimension k and length n using this technique will not work. Moreover, this formula is not in the literature. The size of this intersection would also be needed to determine the degree of every vertex.

Theorem 4.14. *The graph $\Delta_{n,k}$ is a subgraph of $\Gamma_{n,k}$.*

Proof. This is a subgraph since the set of doubly-even codes is a subset of the set of self-orthogonal codes and two codes are connected by an edge in $\Delta_{n,k}$ if and only if they are connected in $\Gamma_{n,k}$. □

We note that $\Delta_{n,k}$ is not a subgraph of $\Theta_{n,k}$ since not all doubly-even codes contain $\mathbf{1}$.

Theorem 4.15. *The number of vertices in $\Delta_{n,1}$ is*

$$\sum_{i=1}^{\lfloor \frac{n}{4} \rfloor} \binom{n}{4i}.$$

Proof. Each dimension 1 doubly-even code consists of the zero vector and a vector with doubly-even weight. This gives the result. □

Example 5. For $n = 1, 2, 3$ the graph $\Delta_{n,k}$ is empty. The graph $\Delta_{4,1}$ has a single vertex and no edges and the graph $\Delta_{4,2}$ is empty. For $n = 5$, $\Delta_{5,1}$ has $\binom{5}{4} = 5$ vertices and they are all connected. Therefore, $\Delta_{5,1}$ is the complete graph on 5 vertices K_5. This gives that there are 10 edges in $\Delta_{5,1}$. The graph $\Delta_{5,2}$ is empty. For $n = 6$, there are $\binom{6}{4} = 15$ vertices. Consider a vector of length 6 and weight

4, for example (111100). To find a weight 4 vector that is not orthogonal to this, it must intersect the support of the vector in 3 places and have exactly one 1 in the remaining 2 places. Therefore, each vertex is connected to $\binom{4}{3}\binom{2}{1} = 8$ vertices. That is $\Delta_{6,1}$ is a regular graph with 15 vertices and degree 8 and therefore has 60 edges.

Example 6. For $n = 7$, there are $\binom{7}{4} = 35$ vertices in $\Delta_{7,1}$. Then, to find a doubly-even vector not orthogonal to a given vector there are $\binom{4}{3}\binom{3}{1} + \binom{4}{1}\binom{3}{3} = 16$ ways to do it. Therefore, the degree of each of the 35 vertices is 16. Therefore, there are 280 edges in the graph.

Given a weight 4 vector of length 7 there are $\binom{4}{2}\binom{3}{2} = 18$ ways of producing a doubly-even vector orthogonal to it. That is, given $\mathbf{v}_1$ there are 18 ways to produce $\mathbf{v}_2$ such that $\mathbf{v}_1$ and $\mathbf{v}_2$ are doubly-even and orthogonal. These generate the code $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2$. Thus, each such code is counted 3 times and we have $\frac{35(18)}{3} = 210$ doubly-even codes of dimension 2 and length 7. That is the number of vertices in $\Delta_{7,2}$ is 210. The degree of every vertex in $\Delta_{7,2}$ is 6 and so there are 630 edges.

Theorem 4.16. *The graph $\Theta_{n,k}$ is regular of degree $2^{n-k} - 2^{n-2k+1}$.*

Proof. There are 2^{n-1} self-orthogonal vectors in the ambient space, that is all of the vectors in $\langle \mathbf{1} \rangle^{\perp}$. Thus, there are $2^{n-1} - 2^{n-k}$ self-orthogonal vectors that are not in the dual of a self-orthogonal code of dimension k. That is, we need a vector $\mathbf{v}$ to be even but not be orthogonal to a code C, since otherwise $\{\mathbf{w} \mid \mathbf{w} \in C,\ [\mathbf{v}, \mathbf{w}] = 0\} = C$ and not a subcode of co-dimension 1. By Lemma 4.3, each code is constructed 2^{k-1} times. This gives that the degree is

$$\frac{2^{n-1} - 2^{n-k}}{2^{k-1}} = 2^{n-k} - 2^{n-2k+1}.$$

□

Example 7. If $k = 1$ then the degree is $2^{n-1} - 2^{n-2+1} = 2^{n-1} - 2^{n-1} = 0$ which coincides with the information in Example 3.

Example 8. If $k = \frac{n}{2}$ then $\Theta_{n,\frac{n}{2}} = \Gamma_n$ where Γ_n is the graph formed by self-dual codes as given in [6], since self-dual codes necessarily contain the all-one vector. Theorem 4.16 gives that the degree of each vertex is $2^{n-k} - 2^{n-2k+1} = 2^{\frac{n}{2}} - 2$ which is the degree given for Γ_n in [6].

We note that Theorem 4.16 would not apply to the graph $\Gamma_{n,k}$ since if the all-one vector were not present in the code C then there are vectors in $C^{\perp}$ that are not self-orthogonal so the counting would not be the same, that is there are not $2^{n-1} - 2^{n-k}$ self-orthogonal vectors that are not in the dual of a self-orthogonal code of dimension k. What we can see is the following.

Theorem 4.17. *If v is a vertex in the graph $\Gamma_{n,k}$ that corresponds to a self-orthogonal code that does not contain the all-one vector then the degree of v is $2^{n-k} - 2^{n-2k}$.*

Proof. There are 2^{n-1} self-orthogonal vectors in the ambient space, that is all of the vectors in $\langle \mathbf{1} \rangle^{\perp}$. Precisely half of the vectors in $C^{\perp}$ are self-orthogonal. That is $D = \{\mathbf{w} \mid [\mathbf{w}, \mathbf{1}] = 0, \mathbf{w} \in C^{\perp}\}$ has co-dimension 1 in $C^{\perp}$. Thus, there are $2^{n-1} - 2^{n-k-1}$ self-orthogonal vectors that are not in the dual of a self-orthogonal code of dimension k. By Lemma 4.3, each code is constructed 2^{k-1} times. This gives that the degree is

$$\frac{2^{n-1}-2^{n-k-1}}{2^{k-1}} = 2^{n-k} - 2^{n-2k}.$$

□

Let us revisit Example 2. The vertices corresponding to the self-orthogonal codes that contain $\mathbf{1}$ have degree 0 as predicted by Theorem 4.16, that is $2^{4-1}-2^{4-2+1} = 2^3-2^3 = 0$ and the vertices corresponding to the self-orthogonal codes that do not contain $\mathbf{1}$ have degree 4 as predicted by Theorem 4.17, that is $2^{4-1}-2^{4-2} = 8-4 = 4$.

Notice then that the graph $\Gamma_{n,k}$ is not regular in general since there are two different degrees a vertex can have.

Theorem 4.18. *If v is a vertex corresponding to a self-orthogonal code containing $\mathbf{1}$ and w is a vertex corresponding to a self-orthogonal code that does not contain $\mathbf{1}$, then v and w are not connected.*

Proof. Let n be even. Let C be the code corresponding to v and D be the code corresponding to w. If they were connected then they would share a subcode E of co-dimension 1. If E contains $\mathbf{1}$ then $E \subset D$ which contradicts the fact that D does not contain $\mathbf{1}$. Therefore, E does not contain $\mathbf{1}$. This means that $C = \langle E, \mathbf{1}\rangle$ and $D = \langle E, \mathbf{u}\rangle$ where $\mathbf{u}$ is some self-orthogonal vector. But this contradicts Theorem 4.6, since $\mathbf{u}$ is self-orthogonal giving that $[\mathbf{u}, \mathbf{1}] = 0$. □

Theorem 4.19. *The graph $\Theta_{n,k}$ and the graph $\Gamma_{n,k} \setminus \Theta_{n,k}$ are connected graphs.*

Proof. Let n be even. Let C and D be self-orthogonal codes that share a subcode E of co-dimension 1 in each. Then $C = \langle E, \mathbf{v}\rangle$ and $D = \langle E, \mathbf{w}\rangle$. If $[\mathbf{v}, \mathbf{w}] \neq 0$ then we are done by Theorem 4.6. If not, then there exists a self-orthogonal vector $\mathbf{u}$ with $[\mathbf{u}, \mathbf{v}] \neq 0$ and $[\mathbf{u}, \mathbf{w}] \neq 0$ as long as none of these vectors is $\mathbf{1}$. Then $N(N(C, \mathbf{u}), \mathbf{w}) = D$. Then applying this at most k times, any two codes are connected in $\Theta_{n,k}$ and in $\Gamma_{n,k} \setminus \Theta_{n,k}$. □

We summarize the results about the graphs.

Theorem 4.20. *Let n be a positive integer and let $k \leq \lfloor \frac{n}{2} \rfloor$.*

1. *If n is odd the number of vertices in $\Gamma_{n,k}$ is $\mathcal{O}_{n,k} = \prod_{i=0}^{k-1} \frac{(2^{n-1-2i}-1)}{(2^{k-i}-1)}$ and the graph $\Gamma_{n,k}$ is a connected regular graph with degree $2^{n-k}-2^{n-2k}$.*
2. *If n is odd then the number of edges in $\Gamma_{n,k} = \prod_{i=0}^{k-1} \frac{(2^{n-1-2i}-1)}{(2^{k-i}-1)}(2^{n-k-1} - 2^{n-2k-1})$.*
3. *If n is even the number of vertices in $\Gamma_{n,k}$ is*
$$\left(2^k \prod_{i=0}^{k-1} \frac{(2^{n-2(i+1)}-1)}{(2^{k-i}-1)}\right) + \left(\prod_{i=0}^{k-2} \frac{(2^{n-2(i+1)}-1)}{(2^{k-i-1}-1)}\right).$$
4. *If n is even, the number of vertices in $\Theta_{n,k}$ is $\frac{\mathcal{H}_{n,k-1}}{2^{k-1}} = \frac{\prod_{i=1}^{k-1}(2^{n-i}-2^i)}{\prod_{i=1}^{k-1}(2^{k-1}-2^{i-1})}\frac{1}{2^{k-1}}$. The degree of each vertex in $\Theta_{n,k}$ is $2^{n-k}-2^{n-2k+1}$. The number of edges in $\Theta_{n,k}$ is*
$$\frac{\prod_{i=1}^{k-1}(2^{n-i}-2^i)}{\prod_{i=1}^{k-1}(2^{k-1}-2^{i-1})}(2^{n-2k}-2^{n-3k+1}).$$

5. *If n is even, the number of vertices in $\Gamma_{n,k} \setminus \Theta_{n,k}$ is $2^k \prod_{i=0}^{k-1} \frac{(2^{n-2(i+1)}-1)}{(2^{k-i}-1)}$ and the degree of each vertex in $\Gamma_{n,k} \setminus \Theta_{n,k}$ is $2^{n-k} - 2^{n-2k}$. The number of edges in $\Gamma_{n,k} \setminus \Theta_{n,k}$ is*
$$\prod_{i=0}^{k-1} \frac{(2^{n-2(i+1)}-1)}{(2^{k-i}-1)}(2^{n-1} - 2^{n-k-1}).$$
6. *If n is even, then $\Gamma_{n,k}$ consists of two disjoint connected subgraphs, $\Theta_{n,k}$ and $\Gamma_{n,k} \setminus \Theta_{n,k}$. Each connected subgraph contains an Euler cycle.*

Proof. The number of vertices in $\Gamma_{n,k}$ are given in Theorem 3.5. If n is odd then $\Theta_{n,k}$ is empty and so $\Gamma_{n,k}$ consists only of vertices corresponding to self-orthogonal codes that do not contain **1**. The degree comes from Theorem 4.16.

If n is odd, then the number of edges is $\frac{1}{2}(\prod_{i=0}^{k-1} \frac{(2^{n-1-2i}-1)}{(2^{k-i}-1)}(2^{n-k} - 2^{n-2k}))$ which gives the result.

If n is even, Lemma 3.3 gives the number of vertices in $\Theta_{n,k}$ and the degree of the vertices comes from Theorem 4.17.

For the number of edges in $\Theta_{n,k}$ we have
$$\begin{aligned}
&\frac{1}{2}\frac{\prod_{i=1}^{k-1}(2^{n-i}-2^i)}{\prod_{i=1}^{k-1}(2^{k-1}-2^{i-1})}\frac{1}{2^{k-1}}(2^{n-k}-2^{n-2k+1})\\
&= \frac{\prod_{i=1}^{k-1}(2^{n-i}-2^i)}{\prod_{i=1}^{k-1}(2^{k-1}-2^{i-1})}\frac{1}{2^{k}}(2^{n-k}-2^{n-2k+1})\\
&= \frac{\prod_{i=1}^{k-1}(2^{n-i}-2^i)}{\prod_{i=1}^{k-1}(2^{k-1}-2^{i-1})}(2^{n-2k}-2^{n-3k+1}).
\end{aligned}$$
For the number of edges in $\Gamma_{n,k} \setminus \Theta_{n,k}$, we have
$$\begin{aligned}
&\frac{1}{2}2^k \prod_{i=0}^{k-1} \frac{(2^{n-2(i+1)}-1)}{(2^{k-i}-1)}(2^{n-k} - 2^{n-2k})\\
&= 2^{k-1} \prod_{i=0}^{k-1} \frac{(2^{n-2(i+1)}-1)}{(2^{k-i}-1)}(2^{n-k} - 2^{n-2k})\\
&= \prod_{i=0}^{k-1} \frac{(2^{n-2(i+1)}-1)}{(2^{k-i}-1)}(2^{n-1} - 2^{n-k-1}).
\end{aligned}$$
If n is even, the degree of every vertex is even, noting that if $k = \frac{n}{2}$, $\Gamma_{n,k} = \Theta_{n,k}$. □

Next, we give a table for the number of edges in the graphs, $\Gamma_{n,k}$, $\Theta_{n,k}$ and $\Gamma_{n,k} \setminus \Theta_{n,k}$. Table 2 gives the number of edges for some small values of n and $k \le \lfloor \frac{n}{2} \rfloor$, by Theorem 4.16. If, for example, $n = 8$ and $k = 3$, we get 315 self-orthogonal codes containing **1**, by Theorem 4.16. We also have that the degree of each vertex in $\Theta_{8,3}$ is $2^{8-3} - 2^{8-6+1} = 24$. Then, the number of edges in $\Theta_{8,3}$ is $\frac{315 \cdot 24}{2} = 3780$, which is also given in the table.

Notice from the table that for even n, and $k = 1$, the graph $\Theta_{n,k}$ has zero edges as the code is generated by the all-one vector and it has no edges. Notice also that for even n, the numbers given for $\Theta_{n,k}$, for $k = \frac{n}{2}$, give the number of edges in the graph $\Gamma_{n,k}$ of self-dual codes. Therefore, the graph $\Gamma_{n,k} \setminus \Theta_{n,k}$ has no edges for those rows as $\Theta_{n,k} = \Gamma_{n,k}$, for $k = \frac{n}{2}$, which is also equal to $\Gamma_{n-1,\lfloor \frac{n}{2} \rfloor}$.

TABLE 2. The number of edges in graphs

$\Gamma_{n,k}$			$\Theta_{n,k}$			$\Gamma_{n,k} \setminus \Theta_{n,k}$		
n	**k**	*#Edges*	**n**	**k**	*#Edges*	**n**	**k**	*#Edges*
1	0	0	2	0	0	2	0	0
3	0	0	2	1	0	2	1	0
3	1	3	4	0	0	4	0	0
5	0	0	4	1	0	4	1	12
5	1	60	4	2	3	4	2	0
5	2	45	6	0	0	6	0	0
7	0	0	6	1	0	6	1	240
7	1	1008	6	2	60	6	2	360
7	2	3780	6	3	45	6	3	0
7	3	945	8	0	0	8	0	0
9	0	0	8	1	0	8	1	4032
9	1	16320	8	2	1008	8	2	30240
9	2	257040	8	3	3780	8	3	15120
9	3	321300	8	4	945	8	4	0
9	4	34425	10	0	0	10	0	0
11	0	0	10	1	0	10	1	65280
11	1	261888	10	2	16320	10	2	2056320
11	2	16695360	10	3	257040	10	3	5140800
11	3	87650640	10	4	321300	10	4	1101600
11	4	46955700	10	5	34425	10	5	0
11	5	2347785	12	0	0	12	0	0
13	0	0	12	1	0	12	1	1047552
13	1	4193280	12	2	261888	12	2	133562880
13	2	$10724 \cdot 10^5$	12	3	16695360	12	3	$14024 \cdot 10^5$
13	3	$22789 \cdot 10^6$	12	4	87650640	12	4	$15026 \cdot 10^5$
13	4	$51276 \cdot 10^6$	12	5	46955700	12	5	150258240
13	5	$12819 \cdot 10^6$	12	6	2347785	12	6	0
13	6	310134825						

We obtain some equalities between the sequences obtained from $\Theta_{n,k}$ and $\Gamma_{n,k}$. In general, we have that $\Theta_{n,k} = \Gamma_{n-1,k-1}$. As a special case, the number sequence obtained by $\Theta_{n,2}$, for even n, is equal to the number sequence obtained by $\Gamma_{n-1,1}$. This sequence is $3, 60, 1008, 16320, 261888, 4193280, \ldots$ and it is given in OEIS in [9], by the reference number A115490 which gives the number of monic irreducible polynomials of degree 4 in $\mathbb{F}_{2^n}[x]$.

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