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THE NUMBER OF CODES OVER RINGS OF ORDER 4 CONTAINING A HULL OF GIVEN TYPE

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Abstract. We study the hull, that is the intersection of a code with its orthogonal, of both linear and additive codes over the rings with unity of order 4, where the orthogonal is the Euclidean orthogonal for linear codes and for additive codes it is determined using characters. We relate the hull of the code with the hull of its image under the corresponding Gray map and use this to count the number of codes with a given hull for additive codes for the rings with characteristic 2. We investigate the codes over these rings which have a hull of given type which gives the cardinality of the hull.

1. **Introduction.** The hull of a linear code is defined to be the intersection of a linear code and its dual with respect to the Euclidean inner-product. This notion was first introduced in [1], by Assmus and Key, and was used to study the codes of finite projective and affine planes. Immediately after this, the hull of a code was also used to study the codes of finite nets in [3] and [4]. In all of these cases, the fact that the hull was a self-orthogonal code was used heavily. Additionally, the hull of a code plays an important role in determining the complexity of algorithms for studying the permutation equivalence of two linear codes and the automorphism group of a linear code. For a complete description of these results, see [20, 21, 27, 26]. It has been shown that these algorithms are very effective if the size of the hull is small. Therefore, finding codes with small hulls has become a widely studied topic of interest in coding theory. In particular, linear complementary dual (LCD) codes, which are codes with a trivial hull have been an area of intense study. See [10] for a description of these codes. Hulls of linear codes, cyclic codes, and dihedral codes over finite fields have been extensively studied in [13], [12], [14], [28], and [30]. Recently, hulls of cyclic codes over the rings $\mathbb{Z}_4$ and $\mathbb{F}_2[v]/\langle v^2+v\rangle$ have been studied in [19] and [18], respectively.

In this paper, we shall study codes over the four finite rings of order 4, namely the rings $\mathbb{F}_4, \mathbb{Z}_4, \mathbb{F}_2[u]/\langle u^2\rangle$, and $\mathbb{F}_2[v]/\langle v^2+v\rangle$. These rings were the first rings to be studied when codes began to be studied over rings. There are several reasons why codes over these rings are important. One of the major reasons, is that there exists interesting Gray maps from these four rings to the binary space. Moreover, these rings represent various different families or rings, namely fields, chain rings,

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and non-chain principal ideal rings. As such, there are generalizations from this family of rings to various other families of rings. In this way, these four rings are representative of codes over rings (at least commutative rings) in general. Additionally, these four rings are all Frobenius, which means they all satisfy the double annihilator condition and that the MacWilliams relations hold for codes over these rings. For a complete description of the importance of codes over Frobenius rings see [5] and [31]. We note that in this paper, we assume that all rings have a multiplicative unity and are commutative. There is a wide literature now for codes over finite commutative rings, but there is now interest in codes over rngs (rings without a multiplicative identity, which gives the spelling, that is "ring" without the "i") and codes over non-commutative rings. However, these cases are quite different than the case we have here so we restrict ourselves to commutative finite rings.

In [25], the number of distinct linear codes over finite fields which have a hull of given dimension were given. It was proven that the expected dimension of the hull of a linear code is a constant when the parameters n and k go to infinity.

In [15], the numbers of distinct linear codes of arbitrary type over a finite chain ring and a finite principal ideal ring were determined. The authors defined a generalized form of the Gaussian binomial coefficients and gave recursion formulas. These formulas will be used extensively here.

In this paper, we define the type of the hull of linear and additive codes over the rings of order 4. We give the number of codes over these rings which have a hull of given type. The main goal in this is to aid in the classification of a certain type of code. For example, LCD codes and self-dual codes are highly sought after codes and the counting formulas are a key step in determining when an exhaustive search for such codes is complete. We describe the hull of linear codes over rings and highlight some nuances of the types of these codes. We show that for three of the rings the hull of the image of a code is the image of the hull of a code. For the ring $\mathbb{Z}_4$, we give a special case where the result is true. For $\mathbb{F}_4$, we determine the number of additive codes with a hull of a given size in Theorem 4.10 and examine the asymptotics when compared to linear codes over $\mathbb{F}_4$ with a give hull size. For $\mathbb{F}_2[v]/\langle v^2+v\rangle$, we first correct some errors in the literature related to the type of a code and we determine the number of additive codes with a hull of a given size. We determine the number of linear codes over this ring and relate this number to the number of binary codes with a given dimension. In Theorem 4.25, we give the number of linear codes over the ring with a hull of a given size. For $\mathbb{F}_2[u]/\langle u^2\rangle$, we count the number of additive codes over the ring. Finally, we count the number of free linear codes over $\mathbb{F}_2[u]/\langle u^2\rangle$ and $\mathbb{Z}_4$ with a hull of a given size.

2. Definitions and notations.

2.1. Codes over rings.

We begin by giving the necessary definitions for rings of order 4 and for codes over these rings.

A code over a ring R of length n, is a subset of R^n. If that subset is also a submodule then we say that the code is linear. If the code is closed under addition but not necessarily closed under scalar multiplication then we say that the code is additive. Of course, all linear codes are additive but not all additive codes are linear. Additive codes over rings of order 4 have been a widely studied object because of their relationship to quantum coding. There is also a natural connection to DNA computing. If a matrix is given as the generator matrix for a linear code then the code is formed by taking all linear combinations of the rows of the matrix. However, if a matrix is given as the generator matrix for an additive code then the code is formed by taking all possible sums of the rows of the matrix.

We attach to the ambient space the standard Euclidean inner-product, namely

$$[\mathbf{v}, \mathbf{w}] = \sum v_i w_i. \tag{1}$$

The orthogonal with respect to this inner product is defined as $C^\perp = \{x \in R^n | [x, y] = 0 \text{ for all } y \in C\}$. It follows from the foundational results in [31] that if R is a finite commutative Frobenius ring then $|C||C^\perp| = |R|^n$, see also [5] for a proof of this result. Of course, the 4 rings we consider in this work are all finite commutative Frobenius rings. A code is said to be self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$. We define the hull of a code C as follows. Let C be a linear code, then

$$Hull(C) = C \cap C^\perp. \tag{2}$$

Since the hull is the intersection of two linear codes, it follows immediately that $Hull(C)$ is a linear code. Moreover, we note that $Hull(C) = Hull(C^\perp)$ since $(C^\perp)^\perp = C$. If $Hull(C) = \{\mathbf{0}\}$, then the code C is said to be a linear code with a complementary dual (LCD), see [10] for a description of these codes and their importance. For a self-orthogonal code C, we have $C = Hull(C)$. With this in mind, a self-dual code has the largest possible hull, namely, $|Hull(C)|^2 = |R|^n$. In other words, the size of the hull is bounded below by 1 and above by $\sqrt{|R|^n}$. Namely, when it is 1 it is an LCD code and when it is $\sqrt{|R|^n}$ it is self-dual.

This Euclidean inner-product given in Equation (1) and its corresponding MacWilliams relations are for linear codes. For additive codes we use the group structure to get a corresponding inner-product.

A character of a group G is a homomorphism from the group G to the multiplicative group of the Complex numbers, $\mathbb{C}^*$. The set of all characters of G, denoted by $\widehat{G} = \{\pi \mid \pi \text{ is a character of } G\}$, is a group that is isomorphic to G, but not in a canonical manner. It is this group isomorphism that we use to construct a duality. An isomorphism between G and its character group $\widehat{G}$ gives a character table as follows. Let $g_1, g_2, \ldots, g_s$ denote the elements of the group and we let $\phi : G \to \widehat{G}$ be the isomorphism. Denote by χ_{g_i} the image of g_i under ϕ, that is $\chi_{g_i} = \phi(g_i)$. Index the rows by $\phi(g_i)$ and the columns by the elements of the groups where the element of the table corresponding to (χ_{g_i}, g_j) is $\chi_{g_i}(g_j)$.

Fix a duality M of G, that is, an isomorphism of G and $\widehat{G}$. Let C be a code over G, then the orthogonal of C with respect to this duality is defined as

$$C^M = \{(g_1, g_2, \ldots, g_n) | \prod_{i=1}^{n} \chi_{g_i}(c_i) = 1 \text{ for all } (c_1, \ldots, c_n) \in C\}. \tag{3}$$

It is known, see [5], that the MacWilliams relations apply for additive codes with this duality. Therefore, we have as consequences the fact that $|C||C^M| = |G|^n$ just as we have for linear codes and their duality. In terms of the hull, we can define the hull with respect to this duality as $Hull_M(C) = C \cap C^M$. It is important to note that if M_1 and M_2 are two different dualities then it is certainly possible (and probable) that $Hull_{M_1}(C) \neq Hull_{M_2}(C^\perp)$.

We recall an important counting result which we shall use extensively. Over the finite field $\mathbb{F}_q$, the number of subcodes of dimension k in a dimension n space is given by a well-known formula,

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

3. **Rings.** In this section, we shall give basic results about the ring of order 4 and give some foundational results about the hull. Let R be a ring of order 4. Note that when we say ring, we are assuming that a ring has a multiplicative unity. Then it is well known that R is one of the following four rings: $\mathbb{F}_4, \mathbb{Z}_4, \mathbb{F}_2[u]/\langle u^2\rangle$, $\mathbb{F}_2[v]/\langle v^2+v\rangle$. The ring $\mathbb{F}_4$ is the field with four elements. The ring $\mathbb{Z}_4$ is the ring of integers modulo 4. We describe each of these rings in the following subsections.

3.1. **The ring $\mathbb{F}_4$.** The ring $\mathbb{F}_4$ is the finite field of order 4. The map α is defined to be the projection from $\mathbb{F}_4$ to $\mathbb{F}_2^2$. That is, $\alpha(a+b\omega) = (a,b)$. Any code over $\mathbb{F}_4$ is equivalent to a code generated by a matrix of the form $(I_k \mid A)$ and has dimension k. It is immediate that the image of any linear code over $\mathbb{F}_4$ under the map α is a binary linear code. However, the inverse image of a binary linear code is necessarily additive but may not be linear. For example, the inverse image of the binary code $\{(0,0),(1,1)\}$ is $\{0, 1+\omega\}$ which is an additive code but not linear. Of course, the image of an additive code under α is a binary linear code.

3.2. **The ring $\mathbb{Z}_4$.** The ring $\mathbb{Z}_4$ is a finite chain ring with maximal ideal $\langle 2\rangle$. The map ϕ is a non-linear map from $\mathbb{Z}_4$ to $\mathbb{F}_2^2$. Writing an element in $\mathbb{Z}_4$ as $a+2b$ we have $\phi(a+2b) = (b, a+b)$. Over $\mathbb{Z}_4$ all additive codes are necessarily linear. The image of a linear code under ϕ be or may not be linear. In fact, it is this Gray map ϕ that was one of the major prompts for studying codes over rings in the first place.

The generator matrix of a code over $\mathbb{Z}_4$ is equivalent to a code generated by a matrix of the following form:

$$G = \begin{pmatrix} I_{k_0} & A_{0,1} & A_{0,2} \\ 0 & 2I_{k_1} & 2A_{1,2} \end{pmatrix}, \tag{4}$$

where $A_{i,j}$ is a binary matrix. In this case, we say that the code has type (k_0, k_1) and has $|C| = 4^{k_0}2^{k_1}$. The dual of the code has type $(n-(k_0+k_1), k_1)$.

3.3. **The ring $\mathbb{F}_2[u]/\langle u^2\rangle$.** The ring $\mathbb{F}_2[u]/\langle u^2\rangle$ is a finite chain ring with maximal ideal $\langle u\rangle$. It differs from $\mathbb{Z}_4$ in that it has characteristic 2 whereas $\mathbb{Z}_4$ has characteristic 4. Codes over this ring were first studied in [7]. This ring is the first in a family of rings denoted by R_k, therefore, this ring is often denoted as R_1. For a description of codes over R_k see [16], [9], and [17]. The map ψ is a linear map from $\mathbb{F}_2[u]/\langle u^2\rangle$ to $\mathbb{F}_2^2$. It is defined as $\psi(a+bu) = (b, a+b)$.

This map is not an isomorphism so that not every binary linear code of even length is the image of a linear code over $\mathbb{F}_2[u]/\langle u^2\rangle$. For example, we note that the action of multiplication by $1+u$ interchanges two coordinates in the binary image. Therefore, the automorphism group of a binary code must contain such an automorphism for the code to be the image of a linear code over $\mathbb{F}_2[u]/\langle u^2\rangle$. We have from Corollary 6.4 in [16] that $\psi(C^\perp) = \psi(C)^\perp$.

The generator matrix of a code over $\mathbb{F}_2[u]/\langle u^2\rangle$ is equivalent to a code generated by a matrix of the following form:

$$G = \begin{pmatrix} I_{k_0} & A_{0,1} & A_{0,2} \\ 0 & uI_{k_1} & uA_{1,2} \end{pmatrix}, \tag{5}$$

where $A_{i,j}$ is a binary matrix. In this case, we say that the code has type (k_0, k_1) and has $|C| = 4^{k_0}2^{k_1}$.

3.4. **The ring $\mathbb{F}_2[v]/\langle v^2+v\rangle$.** The ring $\mathbb{F}_2[v]/\langle v^2+v\rangle$ is isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$ via the Chinese Remainder Theorem. The map β is the inverse of CRT and maps $\mathbb{F}_2[v]/\langle v^2+v\rangle$ linearly to $\mathbb{F}_2^2$. It is defined as $\beta(a+bv) = (a, a+b)$. This map can be extended to $(\mathbb{F}_2[v]/\langle v^2+v\rangle)^n$ in a natural way. Let $C = \beta^{-1}(C_1, C_2)$ be a code over R, then C is denoted by $CRT(C_1, C_2)$, where C_1 and C_2 are binary codes and C is uniquely determined by C_1 and C_2. It is shown in [2] that $\beta(C^\perp) = \beta(C)^\perp$.

It was given in [32] that a linear code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ has a generator matrix of the following form, which consists of a minimal generating set:

$$G = \begin{pmatrix} I_{k_1} & A & B & D_1 + vD_2 \\ 0 & vI_{k_2} & 0 & vC_1 \\ 0 & 0 & (1+v)I_{k_3} & (1+v)E \end{pmatrix},$$

where A, B, C_1, D_1, D_2, E are binary matrices. This is **not true**. Consider the code generated by the vector $(v, 1+v)$. This code is generated by a single vector but does not have a generator matrix of the aforementioned structure. For a complete description see [11]. Therefore, we avoid defining a type for codes over this ring. Rather we deal with the codes and their cardinalities using the Chinese Remainder Theorem.

We now summarize the Gray maps for these 4 rings as follows:

α	ψ	ϕ	β
$\alpha(0) = 00$	$\psi(0) = 00$	$\phi(0) = 00$	$\beta(0) = 00$
$\alpha(1) = 10$	$\psi(1) = 01$	$\phi(1) = 01$	$\beta(1) = 11$
$\alpha(\omega) = 01$	$\psi(1+u) = 10$	$\phi(3) = 10$	$\beta(1+v) = 10$
$\alpha(\omega^2) = 11$	$\psi(u) = 11$	$\phi(2) = 11$	$\beta(v) = 01$

3.5. **The hull of linear codes over rings.** We shall establish some notations for the type of the code and the hull.

Let R be a finite Frobenius ring. Let C be a linear code over R of type $(k_0, k_1, \ldots, k_{e-1})$. Then the hull of C, $Hull(C)$, has type $(l_0, l_1, \ldots, l_{e-1})$. We note for $\mathbb{F}_4$, we have that $e = 1$, for $\mathbb{Z}_4$ and $\mathbb{F}_2[u]/\langle u^2\rangle$ we have that $e = 2$.

We note the following important point. Even though $Hull(C) \subseteq C$, this does not imply that $l_i \leq k_i$. For example, consider the linear code over $\mathbb{Z}_4$, $C = \{00, 11, 22, 33\}$. This code has type $(1, 0)$. Its orthogonal is the code $C^\perp = \{00, 13, 22, 31\}$ which has type $(1, 0)$. The hull of the code, $Hull(C) = \{00, 22\}$, which has type $(0, 1)$. We note that $l_1 = 1$ and $k_1 = 0$ and $1 \not< 0$.

Lemma 3.1. *Let C be a code over a ring R.*

- *If R is $\mathbb{F}_2[u]/\langle u^2\rangle$ then $\psi(C^\perp) = \psi(C)^\perp$.*
- *If R is $\mathbb{F}_2[v]/\langle v^2+v\rangle$ then $\beta(C^\perp) = \beta(C)^\perp$.*
- *If R is $\mathbb{F}_4$ then $\alpha(C^\perp) = \alpha(C)^\perp$.*

Proof. The proof for $\mathbb{F}_2[u]/\langle u^2\rangle$ is found in [17]. The proof for $\mathbb{F}_2[v]/\langle v^2+v\rangle$ is found in [2].

For $\mathbb{F}_4$, let $\mathbf{v} = (a_1 + b_1\omega, a_2 + b_2\omega, \ldots, a_n + b_n\omega)$ and $\mathbf{w} = (c_1 + d_1\omega, c_2 + d_2\omega, \ldots, c_n + d_n\omega)$. Then $[\mathbf{v}, \mathbf{w}] = \sum(a_i + b_i\omega)(c_i + d_i\omega)$. If $[\mathbf{v}, \mathbf{w}] = 0$, then $\sum(a_ic_i + b_id_i) + (b_ic_i + a_id_i + b_id_i)\omega = 0$. This implies that $\sum(a_ic_i + b_id_i) = 0$. Then $\alpha(\mathbf{v}) = (a_1, b_2, a_2, b_2, \ldots, a_n, b_n)$ and $\alpha(\mathbf{w}) = (c_1, d_2, c_2, d_2, \ldots, c_n, d_n)$. Therefore,

$[\alpha(\mathbf{v}), \alpha(\mathbf{w})] = \sum a_i c_i + b_i d_i$. This gives that $[\mathbf{v}, \mathbf{w}] = 0$ implies $[\alpha(\mathbf{v}), \alpha(\mathbf{w})] = 0$. Then noting that $|\alpha(C)^\perp| = |\alpha(C^\perp)|$ and we have the result. □

Theorem 3.2. *Let C be a code of length n over $\mathbb{F}_2[u]/\langle u^2\rangle$, $\mathbb{F}_2[v]/\langle v^2+v\rangle$, and $\mathbb{F}_4$ and let Φ be its corresponding Gray map. Then*

$$\Phi(Hull(C)) = Hull(\Phi(C)).$$

Proof. Let $\mathbf{v} \in Hull(\Phi(C))$. Then $\mathbf{v} \in \Phi(C) \cap \Phi(C)^\perp = \Phi(C) \cap \Phi(C^\perp)$. Let $\mathbf{w}$ be such that $\Phi(\mathbf{w}) = \mathbf{v}$. Then $\Phi(\mathbf{w}) \in \Phi(C)$ and $\Phi(\mathbf{w}) \in \Phi(C^\perp)$. This gives $\mathbf{w} \in C \cap C^\perp$ and $\mathbf{v} = \Phi(\mathbf{w}) \in \Phi(C \cap C^\perp) = \Phi(Hull(C))$. Therefore, $Hull(\Phi(C)) \subseteq \Phi(Hull(C))$.

Next let $\mathbf{v} \in \Phi(Hull(C)) = \Phi(C \cap C^\perp)$. Let $\mathbf{w}$ be such that $\Phi(\mathbf{w}) = \mathbf{v}$. Then $\mathbf{w} \in C \cap C^\perp = Hull(C)$. This gives that $\mathbf{v} \in \Phi(C)$ and $\mathbf{v} \in \Phi(C^\perp) = \Phi(C)^\perp$. Then $\mathbf{v} \in \Phi(C) \cap \Phi(C)^\perp = Hull(\Phi(C))$. This gives $\Phi(Hull(C)) \subseteq Hull(\Phi(C))$ and we have the result. □

For $\mathbb{Z}_4$ we do not have the result in the previous theorem since the Gray map does not satisfy the condition that $\phi(C^\perp) = \phi(C)^\perp$. See [6] for an explanation where it gives numerous examples of self-dual codes whose images are not self-dual (recall that the hull of a self-dual code is the code itself). We shall give a simple example of why it fails for $\mathbb{Z}_4$ in Example 1. What we do have in this case is given in the following theorem.

Theorem 3.3. *Let C be a code over $\mathbb{Z}_4$ of type $(0, k_1)$. Then*

$$\phi(Hull(C)) = Hull(\phi(C)).$$

Proof. Any code over $\mathbb{Z}_4$ of this type is necessarily self-orthogonal. Therefore $Hull(C) = C$ so $\phi(Hull(C)) = \phi(C)$.

It is clear that $\phi(C)$ is also self-orthogonal so $Hull(\phi(C)) = \phi(C)$ and we have the equality. □

Example 1. Consider the linear code over $\mathbb{Z}_4$, $C = \{(0,0),(1,1),(2,2),(3,3)\}$. Its orthogonal is $C^\perp = \{(0,0),(1,3),(2,2),(3,1)\}$. Then we have that $Hull(C) = \{(0,0),(2,2)\}$. The Gray image of the linear code C is $\phi(C) = \{(0,0,0,0),(0,1,0,1),(1,1,1,1),(1,0,1,0)\}$. This binary code is self-orthogonal (in fact it is self-dual), so $Hull(\phi(C)) = \phi(C)$. It is immediate that $\phi(Hull(C)) \neq Hull(\phi(C))$. This shows that it is not always the case for codes over $\mathbb{Z}_4$ that $\phi(Hull(C)) = Hull(\phi(C))$.

We note that in this case $Hull(\phi(C^\perp)) = \phi(C^\perp)$ as well and so $\phi(Hull(C^\perp)) \neq Hull(\phi(C^\perp))$.

4. **Counting codes.** In this section, we begin by giving some foundational counting results which we shall use to determine the number of codes with a hull of a given size.

In [25], the number of distinct linear codes over finite fields which have a hull of given dimension and the average hull dimension of linear codes was given. We consider here the case for the rings of order 4, which is the starting point when studying codes over rings. The ring $\mathbb{F}_4$ is the field with 4 elements so it was studied in [25].

Our main goal is to answer the following question. How many linear (additive) codes C over a ring R are there where $|Hull(C)|$ is a given constant. Of course, there are only certain possible values for this constant. The problem can be made more precise when we have a definition of type. Then we can say, how many codes are there where the type of the $Hull(C)$ is (k_0, k_1). We can restate the problem according to the following theorem.

Theorem 4.1. *The number of linear (additive) codes C of length n over a finite commutative Frobenius ring R with $|Hull(C)| = m$ is equal to the number of linear codes C over R with $|C + C^\perp| = \frac{|R|^n}{m}$.*

Proof. Given a linear code C, we have, $C \subset Hull(C)^\perp$ and $C^\perp \subset Hull(C)^\perp$. Then

$$\begin{aligned} |C + C^\perp| &= \frac{|C||C^\perp|}{|Hull(C)|} \\ &= \frac{|R|^n}{|Hull(C)|} \\ &= |Hull(C)^\perp|. \end{aligned}$$

This gives the result.

The same proof is true for additive codes by simply replacing $C^\perp$ with C^M. $\square$

We note that for linear codes over fields, we get the result about the cardinality of the code $C + C^\perp$ by considering dimensions of codes, but we do not have the notion of a dimension for a module over a finite commutative ring. This means that we can phrase the question in terms of the size of the intersection or in terms of the size of the sum of the codes.

We give the following theorem for the rings of order 4 as a corollary of Theorem 3.3 in the authors' paper [15], which gives the number of distinct codes over a finite chain ring with maximal ideal $\langle\gamma\rangle$, where γ has nilpotency e. We require the following results to perform the desired counting. The first counts the number of linear codes over the chain rings of order 4.

Theorem 4.2. [15] *Let R be one of the rings $\mathbb{Z}_4$ or $\mathbb{F}_2 + u\mathbb{F}_2$, $u^2 = 0$, with type (k_0, k_1), with maximal ideal $|R/m| = q = 2$ and nilpotency $e = 2$. The number of linear codes over R is given by the following formula,*

$$\begin{bmatrix} n \\ k_0, k_1 \end{bmatrix}_{q,e} = \begin{bmatrix} n \\ k_0, k_1 \end{bmatrix}_{2,2} = \frac{\prod_{i=0}^{k_0-1}(2^{2n} - 2^{n+i})\prod_{j=0}^{k_1-1}(2^n - 2^{k_0+j})}{\prod_{i=0}^{k_0-1}(2^{2k_0+k_1} - 2^{k_0+k_1+i})\prod_{j=0}^{k_1-1}(2^{k_0+k_1} - 2^{k_0+j})}.$$

The second is the following theorem from [22, 23, 24], where they count the number of self-orthogonal codes over a finite field.

Theorem 4.3. [22, 23, 24] *Let n and q be positive even integers and $k \le n/2$. The number of self-orthogonal codes over $\mathbb{F}_q$ of length n and dimension k is*

$$\sigma_{n,k} = \frac{q^{n-k} - 1}{q^n - 1}\prod_{i=1}^{k}\frac{q^{n-2i+2} - 1}{q^i - 1}.$$

The third and fourth results are the following two lemmas given in [25], for codes over finite fields. Note that the first lemma counts subcodes whereas the second counts supercodes.

Lemma 4.4. [25] *Let C be a linear code over $\mathbb{F}_q$ of length n and dimension k. The number of self-orthogonal codes V over $\mathbb{F}_q$ of length n and dimension l such that $V \subseteq Hull(C)$ is the Gaussian binomial $\begin{bmatrix} dim(Hull(C)) \\ l \end{bmatrix}_q$.*

Lemma 4.5. [25] *Let V be a binary self-orthogonal code over $\mathbb{F}_q$ of length n and dimension l. The number of binary linear codes C over $\mathbb{F}_q$ of length n and dimension k such that $V \subseteq Hull(C)$ is $\begin{bmatrix} n-2l \\ k-l \end{bmatrix}_q$.*

The final counting result we need is the following theorem which is Theorem 4.5 in [25], when $q = 2$.

Theorem 4.6. [25] *Let $\sigma_{n,i}$ denote the number of self-orthogonal codes over $\mathbb{F}_q$ of length n and dimension i. Let $k \leq n/2$ and $l \leq k$. The number of linear codes over $\mathbb{F}_q$ of length n and dimension k where the dimension of the hull is l is*

$$\sum_{i=l}^{k} \begin{bmatrix} n-2i \\ k-i \end{bmatrix}_q \begin{bmatrix} i \\ l \end{bmatrix}_q (-1)^{i-l} q^{\binom{i-l}{2}} \sigma_{n,i}.$$

4.1. **Counting codes over $\mathbb{F}_4$.** We now shall count codes over $\mathbb{F}_4$, both codes that are additive and codes that are linear. Consider the following duality on the additive group of $\mathbb{F}_4$:

M_E	0	1	ω	$1+\omega$
0	1	1	1	1
1	1	-1	1	-1
ω	1	1	-1	-1
$1+\omega$	1	-1	-1	1

Lemma 4.7. *Let $\mathbf{v}, \mathbf{w} \in \mathbb{F}_4^n$. Then $[\mathbf{v}, \mathbf{w}]_M = 1$ if and only if $[\alpha(\mathbf{v}), \alpha(\mathbf{w})] = 0$.*

Proof. The result follows by examining the table of inner-products of the image of the elements of $\mathbb{F}_4$, under α.

	00	10	01	11
00	0	0	0	0
10	0	1	0	1
01	0	0	1	1
11	0	1	1	0

□

Theorem 4.8. *Let C be an additive code over $\mathbb{F}_4$. Then $Hull(\alpha(C)) = \alpha(Hull_{M_E}(C))$.*

Proof. The proof is the same as Theorem 3.2 using Lemma 4.7. □

This leads to the following important theorem for additive codes over $\mathbb{F}_4$.

Theorem 4.9. *The number of additive codes over $\mathbb{F}_4$ of length n where $Hull_{M_E}(C)$ has size 2^k is equal to the number of binary linear codes of length $2n$ with hulls of dimension k.*

Proof. Every additive code over $\mathbb{F}_4$ of length n is of the form $\alpha^{-1}(\mathcal{C})$ for some binary linear code $\mathcal{C}$ of length $2n$. Then by Theorem 4.8 we have the result. □

This brings us to the most important result for additive codes where we count the number of additive codes with a hull of given size.

Theorem 4.10. *The number of additive codes over* $\mathbb{F}_4$ *of length* n *and size* 2^k *whose hull with respect to* M_E *has size* 2^l *with* $l \le k$ *and* $k \le n$ *is*

$$\sum_{i=l}^{k} \begin{bmatrix} 2n-2i \\ k-i \end{bmatrix}_2 \begin{bmatrix} i \\ l \end{bmatrix}_2 (-1)^{i-l} 2^{\binom{i-l}{2}} \sigma_{2n,i},$$

where $\sigma_{n,i}$ *is the number of binary self-orthogonal codes of length* n *and dimension* i*.*

Proof. The result follows from Theorem 4.9 and Theorem 4.6. □

Example 2. Let $n = 2$, $k = 2$ and $l = 1$. Then by Theorem 4.3, $\sigma_{4,1} = 7$ and $\sigma_{4,2} = 3$. The number of additive codes over $\mathbb{F}_4$ of length 2 and size 2^2 whose hull has size 2^1 is 12 by Theorem 4.10. These are the additive codes given by the following generating matrices:

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ w & w \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1+w \end{pmatrix}, \begin{pmatrix} w & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} w & 0 \\ 1 & w \end{pmatrix}, \begin{pmatrix} w & 0 \\ 0 & 1+w \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1+w & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & w \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ w & w \end{pmatrix}, \begin{pmatrix} 0 & w \\ 1+w & 0 \end{pmatrix}, \begin{pmatrix} 0 & w \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & w \\ w & 1 \end{pmatrix}.$$

Note that Theorem 4.6 gives the number of linear codes over $\mathbb{F}_4$ of length n and dimension k where the dimension of the hull is l.

Theorem 4.11. *The ratio of the number of linear codes over* $\mathbb{F}_4$ *of length* n *and dimension* k *and the number of additive codes over* $\mathbb{F}_4$ *of length* n *and size* 4^k *goes to* 0 *as* n *goes to infinity.*

Proof. The number of linear codes of length n and dimension k is $\begin{bmatrix} n \\ k \end{bmatrix}_4$ and the number of additive codes of length n and size 4^k is the same as the number of binary linear codes of length $2n$ and dimension $2k$ which is $\begin{bmatrix} 2n \\ 2k \end{bmatrix}_2$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_4}{\begin{bmatrix} 2n \\ 2k \end{bmatrix}_2} &= \frac{\frac{(4^n-1)(4^{n-1}-1)(4^{n-2}-1)\cdots(4^{n-k+1}-1)}{(4^k-1)(4^{k-1}-1)\cdots(4-1)}}{\frac{(2^{2n}-1)(2^{2n-1}-1)\cdots(2^{2n-2k+1}-1)}{(2^{2k}-1)(2^{2k-1}-1)\cdots(2-1)}} \\
&= \lim_{n\to\infty} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_4}{\begin{bmatrix} n \\ k \end{bmatrix}_4 \frac{(2^{2n-1}-1)(2^{2n-3}-1)\cdots(2^{2n-2k+1}-1)}{(2^{2k-1}-1)(2^{2k-3}-1)\cdots(2-1)}} \\
&= \lim_{n\to\infty} \frac{1}{\frac{(2^{2n-1}-1)(2^{2n-3}-1)\cdots(2^{2n-2k+1}-1)}{(2^{2k-1}-1)(2^{2k-3}-1)\cdots(2-1)}} \\
&= 0.
\end{aligned}$$

□

This result gives another reason for the importance of additive codes. Namely, they are abundant with respect to linear codes of the same size.

Theorem 4.12. *Let N_1 be the number of linear codes over $\mathbb{F}_4$ of length n and dimension k with a given hull of dimension l. Let N_2 be the number of additive codes over $\mathbb{F}_4$ of length n and size 4^k whose hull with respect to M_E has size 4^l, $l \leq k$. Then the ratio $\frac{N_1}{N_2}$ goes to 0 as n goes to infinity.*

Proof. By Theorem 4.6, when $q = 4$, we have

$$N_1 = \sum_{i=l}^{k} \begin{bmatrix} n-2i \\ k-i \end{bmatrix}_4 \begin{bmatrix} i \\ l \end{bmatrix}_4 (-1)^{i-l} 4^{\binom{i-l}{2}} \sigma_{n,i},$$

where $\sigma_{n,i}$ denotes the number of self-orthogonal codes over $\mathbb{F}_4$ of length n and dimension i, $i = l, \ldots, k$. By Theorem 4.10, we have

$$N_2 = \sum_{j=l}^{k} \begin{bmatrix} 2n-4j \\ 2k-2j \end{bmatrix}_2 \begin{bmatrix} 2j \\ 2l \end{bmatrix}_2 (-1)^{2j-2l} 2^{\binom{2j-2l}{2}} \sigma_{2n,2j},$$

where $\sigma_{2n,2j}$ denotes the number of binary self-orthogonal codes of length $2n$ and dimension $2j$, $j = l, \ldots, k$.

Note that we are interested in the additive codes over $\mathbb{F}_4$ of length n which have a size of 4^k and so they are the binary linear codes of length $2n$ and dimension 2^{2k}.

First, we have the following equalities:

1. $\dfrac{\begin{bmatrix} n-2s \\ k-s \end{bmatrix}_4}{\begin{bmatrix} 2n-4s \\ 2k-2s \end{bmatrix}_2} = \frac{1}{\frac{(2^{2n-4s-1}-1)(2^{2n-4s-3}-1)\cdots(2^{2n-2s-2k+1}-1)}{(2^{2k-2s-1}-1)(2^{2k-2s-3}-1)\cdots(2-1)}}$.
2. $\dfrac{\begin{bmatrix} s \\ l \end{bmatrix}_4}{\begin{bmatrix} 2s \\ 2l \end{bmatrix}_2} = \frac{1}{\frac{(2^{2s-1}-1)(2^{2s-3}-1)\cdots(2^{2s-2l+1}-1)}{(2^{2l-1}-1)(2^{2l-3}-1)\cdots(2-1)}}$
3. $\frac{(-1)^{s-l}}{(-1)^{2s-2l}} = (-1)^{s/2+l}$.
4. $\frac{4^{\binom{s-l}{2}}}{2^{\binom{2s-2l}{2}}} = 2^{-2(l-s)(l+s)}$.
5. $\frac{\sigma_{n,s}}{\sigma_{2n,2s}} = \frac{1}{\frac{(2^{2n-2}-1)(2^{2n-6}-1)\cdots(2^{2n-4s+2}-1)}{(2^{2s-1}-1)(2^{2s-3}-1)\cdots(2-1)}}$.

For $j = 1, 2 \ldots, k$, we rewrite the fractions given above as follows:

1. $\begin{bmatrix} 2n-4j \\ 2k-2j \end{bmatrix}_2 \cdot \frac{1}{\frac{(2^{2n-4j-1}-1)(2^{2n-4j-3}-1)\cdots(2^{2n-2j-2k+1}-1)}{(2^{2k-2j-1}-1)(2^{2k-2j-3}-1)\cdots(2-1)}}$, for $\begin{bmatrix} n-2j \\ k-j \end{bmatrix}_4$,
2. $\begin{bmatrix} 2j \\ 2l \end{bmatrix}_2 \cdot \frac{1}{\frac{(2^{2j-1}-1)(2^{2s-3}-1)\cdots(2^{2j-2l+1}-1)}{(2^{2l-1}-1)(2^{2l-3}-1)\cdots(2-1)}}$, for $\begin{bmatrix} j \\ l \end{bmatrix}_4$,
3. $(-1)^{2j-2l} \cdot (-1)^{j/2+2}$ for $(-1)^{j-l}$.
4. $2^{\binom{2j-2l}{2}} \cdot 2^{-2(l-j)(l+j)}$, for $4^{\binom{j-l}{2}}$
5. $\sigma_{2n,2j} \cdot \frac{1}{\frac{(2^{2n-2}-1)(2^{2n-6}-1)\cdots(2^{2n-4s+2}-1)}{(2^{2j-1}-1)(2^{2j-3}-1)\cdots(2-1)}}$, for $\sigma_{n,j}$.

Finally, we use the last statements to get the result:

$$\lim_{n\to\infty} \frac{N_1}{N_2}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{\sum_{i=l}^{k} \begin{bmatrix} n-2i \\ k-i \end{bmatrix}_4 \begin{bmatrix} i \\ l \end{bmatrix}_4 (-1)^{i-l} 4^{\binom{i-l}{2}} \sigma_{n,i}}{\sum_{j=l}^{k} \begin{bmatrix} 2n-4j \\ 2k-2j \end{bmatrix}_2 \begin{bmatrix} 2j \\ 2l \end{bmatrix}_2 (-1)^{2j-2l} 2^{\binom{2j-2l}{2}} \sigma_{2n,2j}} \\
&= \lim_{n\to\infty} \frac{\sum_{j=l}^{k} \begin{bmatrix} 2n-4j \\ 2k-2j \end{bmatrix}_2 \cdot \frac{1}{\frac{(2^{2n-4j-1}-1)(2^{2n-4j-3}-1)\cdots(2^{2n-2j-2k+1}-1)}{(2^{2k-2j-1}-1)(2^{2k-2j-3}-1)\cdots(2-1)}} \cdot \sigma_{2n,2j}}{\sum_{j=l}^{k} \begin{bmatrix} 2n-4j \\ 2k-2j \end{bmatrix}_2 \begin{bmatrix} 2j \\ 2l \end{bmatrix}_2 (-1)^{2j-2l} 2^{\binom{2j-2l}{2}} \sigma_{2n,2j}} \\
&\quad \cdot \frac{1}{\frac{(2^{2n-2}-1)(2^{2n-6}-1)\cdots(2^{2n-4j+2}-1)}{(2^{2j-1}-1)(2^{2j-3}-1)\cdots(2-1)}} \cdot \left((-1)^{2j-2l} (-1)^{j/2+2} 2^{\binom{2j-2l}{2}} 2^{-2(l-j)(l+j)} \right) \cdot \begin{bmatrix} 2j \\ 2l \end{bmatrix}_2 \\
&\quad \cdot \frac{1}{\frac{(2^{2j-1}-1)(2^{2j-3}-1)\cdots(2^{2j-2l+1}-1)}{(2^{2l-1}-1)(2^{2l-3}-1)\cdots(2-1)}} \\
&= 0.
\end{aligned}
$$

□

4.2. **Counting codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$.** We now move to the ring $\mathbb{F}_2[v]/\langle v^2+v\rangle$ where we shall use the Chinese Remainder Theorem extensively in our counting. We shall begin by counting additive codes, then linear codes, then self-orthogonal codes. These are required to count the number of codes with a given hull size.

Consider the following duality on the additive group of $\mathbb{F}_2[v]/\langle v^2+v\rangle$:

M_E	0	1	v	$1+v$
0	1	1	1	1
1	1	1	-1	-1
v	1	-1	-1	1
$1+v$	1	-1	1	-1

Lemma 4.13. *Let $\mathbf{v}, \mathbf{w} \in \mathbb{F}_2[v]/\langle v^2+v\rangle^n$. Then $[\mathbf{v},\mathbf{w}]_M = 1$ if and only if $[\beta(\mathbf{v}), \beta(\mathbf{w})] = 0$.*

Proof. The result follows by examining the table of inner-products of the image of the elements of $\mathbb{F}_2[v]/\langle v^2+v\rangle$, under β.

	0	11	01	10
0	0	0	0	0
11	0	0	1	1
01	0	1	1	0
10	0	1	0	1

□

Theorem 4.14. *Let C be an additive code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$. Then $Hull(\beta(C)) = \beta(Hull_{M_e}(C))$.*

Proof. The proof is the same as Theorem 3.2 using Lemma 4.13. □

This leads to the following important theorem for additive codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$.

Theorem 4.15. *The number of additive codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n where $Hull_{M_E}(C)$ has size 2^k is equal to the number of binary linear codes of length $2n$ with hulls of dimension k.*

Proof. Every additive code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n is of the form $\beta^{-1}(\mathcal{C})$ for some linear binary code $\mathcal{C}$ of length $2n$. Then by Theorem 4.14 we have the result. □

Theorem 4.16. *The number of additive codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n and size 2^k whose hull size is 2^l, $l \le k$, is*

$$\sum_{i=l}^{k} \begin{bmatrix} 2n-2i \\ k-i \end{bmatrix}_2 \begin{bmatrix} i \\ l \end{bmatrix}_2 (-1)^{i-l} 2^{\binom{i-l}{2}} \sigma_{2n,i}.$$

Proof. The result follows from Theorem 4.6 and Theorem 4.15. □

Example 3. Let $n=2$, $k=2$ and $l=1$. Then by Theorem 4.3, $\sigma_{4,1}=7$ and $\sigma_{4,2}=3$. The number of additive codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$, $v^2=v$, of length 2 and size 2^2 whose hull has size 2^1 is 12 by Theorem 4.16. These are the additive codes given by the following generating matrices:

$$\begin{pmatrix} 1+v & 0 \\ v & 1+v \end{pmatrix}, \begin{pmatrix} 1+v & 0 \\ v & v \end{pmatrix}, \begin{pmatrix} 1+v & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} v & 0 \\ 1+v & 1+v \end{pmatrix}, \begin{pmatrix} v & 0 \\ 1+v & v \end{pmatrix},$$

$$\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1+v \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1+v \\ 1+v & v \end{pmatrix}, \begin{pmatrix} 0 & 1+v \\ v & v \end{pmatrix}, \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & v \\ 1+v & 1+v \end{pmatrix}, \begin{pmatrix} 0 & v \\ v & 1+v \end{pmatrix}.$$

Next we shall count the linear codes in this setting. We begin with a technical lemma.

Lemma 4.17. *Let $C = CRT(C_1, C_2)$ be a linear code over the ring $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n and size 2^k. The Gray image $\beta(C)$ of C is a binary linear code of length $2n$ and dimension $k = k_1 + k_2$, where C_i is a binary linear code of length n and dimension k_i, for $i = 1, 2.$ respectively.*

Proof. It follows from the fact that β is an isomorphism between the rings $\mathbb{F}_2[v]/\langle v^2+v\rangle$ and $\mathbb{F}_2 \times \mathbb{F}_2$. □

Following definition is given in [32].

Definition 4.18. Let $\mathbf{x} = (x_1, \ldots, x_n) \in (\mathbb{F}_2[v]/\langle v^2+v\rangle)^n$ with $x_i = r_i + vq_i$, with $1 \le i \le n$.

$$\beta(\mathbf{x}) = (\mathbf{r}, \mathbf{r}+\mathbf{q}),$$

such that $\mathbf{x} = \mathbf{r} + v\mathbf{q} \in (\mathbb{F}_2[v]/\langle v^2+v\rangle)^n$ where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ are binary vectors.

By the previous definition, we have the following important theorem. It is also given in [32].

Theorem 4.19. *Let C be a linear code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n. Then $\beta(C)$ can be written as a direct product of binary linear codes C_1 and C_2 where $C_1 = \{\mathbf{x} \in \mathbb{F}_2^n : \mathbf{x} + v\mathbf{y} \in C, \mathbf{y} \in \mathbb{F}_2^n\}$ and $C_2 = \{\mathbf{x}+\mathbf{y} \in \mathbb{F}_2^n : \mathbf{x} + v\mathbf{y} \in C\}$. Moreover, $C = CRT(C_1, C_2)$.*

Proof. Let C be a linear code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$. Let $\beta(C)$ be the image of C under β. Take a vector $\mathbf{w} = (r_1, \ldots, r_n, q_1, \ldots, q_n) \in \beta(C)$. Let $c_i = r_i + v(r_i + q_i)$, $i = 1, 2\ldots, n$. Then, since β is a bijection, $\beta^{-1}(\mathbf{w}) = \mathbf{c} = (c_1, c_2, \ldots, c_n) \in C$. Let $C_1 = \{\mathbf{x} \in \mathbb{F}_2^n | \mathbf{x} + v\mathbf{y} \in C, \mathbf{y} \in \mathbb{F}_2^n\}$ and $C_2 = \{\mathbf{x}+\mathbf{y} \in \mathbb{F}_2^n | \mathbf{x} + v\mathbf{y} \in C\}$. Then we get $(r_1, \ldots, r_n, q_1, \ldots, q_n) \in C_1 \times C_2$, where $(r_1, \ldots, r_n) \in C_1$ and $(q_1, \ldots, q_n) \in C_2$. Therefore, we have that $\beta(C) \subseteq C_1 \times C_2$. On the other hand, take $(r_1, \ldots, r_n, q_1, \ldots, q_n) \in C_1 \times C_2$. Then $(r_1, \ldots, r_n) \in C_1$ and $(q_1, \ldots, q_n) \in C_2$.

Then by the definitions of C_1 and C_2, we have that $c = (c_1, c_2, \dots, c_n)$, where $c_i = r_i + v(r_i + q_i)$. Then $\mathbf{c} = \mathbf{r} + v(\mathbf{r} + \mathbf{q})$, and so $\beta(\mathbf{c}) = (r_1, \dots, r_n, q_1, \dots, q_n) \in \beta(C)$, which means that $C_1 \times C_2 \subseteq \beta(C)$. Therefore we have that $\beta(C) = C_1 \times C_2$. By the definitions of C_1 and C_2, we have that $CRT(C_1, C_2) = C$. □

We first need to determine the number of linear codes since this was not previously given in the literature.

Theorem 4.20. *The number of linear codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n and size 2^k is equal to*

$$\sum_{k_1+k_2=k} \begin{bmatrix} n \\ k_1 \end{bmatrix}_2 \cdot \begin{bmatrix} n \\ k_2 \end{bmatrix}_2 .$$

Proof. Let C be a linear code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ such that $C = \beta^{-1}(C_1, C_2)$, where C_1 and C_2 are binary linear codes of length n and dimension k_1 and k_2, respectively. A linear code C over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n of size 2^k is uniquely determined by two binary linear codes C_1 and C_2, both of length n and dimension k_1 and k_2, respectively. Then the product of the number of binary linear codes C_1, $\begin{bmatrix} n \\ k_1 \end{bmatrix}_2$, and the number of binary linear codes C_2, $\begin{bmatrix} n \\ k_2 \end{bmatrix}_2$, for each $k = k_1 + k_2$, gives the desired result. □

Example 4. Let $n = 2$ and $k = 2$. The number of linear codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n and size 2^2 is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}_2 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}_2 + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 + \begin{bmatrix} 2 \\ 2 \end{bmatrix}_2 \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix}_2 = 1 + 9 + 1 = 11$. These codes are given by the following generating matrices:

$$\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & v \end{pmatrix}, \begin{pmatrix} 1 & 1+v \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} v & 1 \end{pmatrix}, \begin{pmatrix} 1+v & 1 \end{pmatrix},$$

$$\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & 1+v \end{pmatrix}, \begin{pmatrix} 1+v & 0 \\ 0 & v \end{pmatrix}, \begin{pmatrix} 1+v & 0 \\ 0 & 1+v \end{pmatrix}.$$

The following lemma is given in [8].

Lemma 4.21. *Let C be a linear code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n. Then $C = CRT(C_1, C_2)$ is Euclidean self-orthogonal if and only if C_1 and C_2 are binary self-orthogonal codes.*

Next, we need to determine the number of self-orthogonal codes.

Lemma 4.22. *Let C be a linear code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n and size 2^k. The number of Euclidean self-orthogonal codes V over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n and size 2^l such that $V \subseteq Hull(C)$ is equal to the number*

$$\sum_{l_1+l_2=l} \begin{bmatrix} dim(Hull(C_1)) \\ l_1 \end{bmatrix} \cdot \begin{bmatrix} dim(Hull(C_2)) \\ l_2 \end{bmatrix},$$

where $C = CRT(C_1, C_2)$, C_1 and C_2 are binary linear codes and l_i is the dimension of the binary self-orthogonal code $V_i \subseteq Hull(C_i)$, $l_i \leq dim(Hull(C_i))$, for $i = 1, 2$, such that $l_1 + l_2 = l$ and $CRT(V_1, V_2) = V$.

Proof. Let C be a linear code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n such that $C = CRT(C_1, C_2)$, where C_1 and C_2 are binary linear codes both of length n and dimension k_1 and k_2, respectively. Let $\mathbf{c} = \mathbf{r} + v\mathbf{q}$ be a codeword in C, then by

Definition 4.18, we have that $\beta(\mathbf{c}) = (\mathbf{r}, \mathbf{r}+\mathbf{q})$, where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ are binary vectors. By Theorem 4.19, we write $\beta(C) = C_1 \times C_2$, where $C_1 = \{\mathbf{x} \in \mathbb{F}_2^n | \mathbf{x} + v\mathbf{y} \in C, y \in \mathbb{F}_2^n\}$ and $C_2 = \{\mathbf{x}+\mathbf{y} \in \mathbb{F}_2^n | \mathbf{x} + v\mathbf{y} \in C\}$. Then, we have that

$$\begin{aligned} Hull(\beta(C)) &= \beta(C) \cap (\beta(C))^{\perp} \\ &= C_1 \times C_2 \cap (C_1 \times C_2)^{\perp} \\ &= C_1 \times C_2 \cap C_1^{\perp} \times C_2^{\perp} \\ &= C_1 \cap C_1^{\perp} \times C_2 \cap C_2^{\perp} \\ &= Hull(C_1) \times Hull(C_2). \end{aligned}$$

By Theorem 3.2, we have that $\beta(Hull(C)) = Hull(\beta(C))$ and so $\beta(Hull(C)) = Hull(C_1) \times Hull(C_2)$.

On the other hand, let $V = CRT(V_1, V_2)$ be a Euclidean self orthogonal code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ such that $V \subseteq Hull(C)$. By Lemma 4.21, we have that V_1 and V_2 are binary self-orthogonal codes of dimension l_1 and l_2, respectively, such that $l_1 + l_2 = l$. By Theorem 4.19, we have that $\beta(V) = V_1 \times V_2$, where $V_1 = \{\mathbf{x} \in \mathbb{F}_2^n | \mathbf{x} + v\mathbf{y} \in V, y \in \mathbb{F}_2^n\}$ and $V_2 = \{\mathbf{x}+\mathbf{y} \in \mathbb{F}_2^n | \mathbf{x} + \mathbf{v}y \in V\}$. Take a vector $\mathbf{w} \in \beta(V)$. Then since β is a bijection, we have $\beta^{-1}(\mathbf{w}) \in V \subseteq Hull(C)$. Therefore, we have that $\mathbf{w} \in \beta(Hull(C))$, which means that $\beta(V) \subseteq \beta(Hull(C)) = Hull(C_1) \times Hull(C_2)$. Therefore, we have that $V_1 \times V_2 = \beta(V) \subseteq Hull(C_1) \times Hull(C_2)$. So we have that $V_1 \subseteq Hull(C_1)$ and that $V_2 \subseteq Hull(C_2)$, by noting that all the linear codes have at least the zero vector so that they are not empty. The number of binary self-orthogonal codes V_1 such that $V_1 \subseteq Hull(C_1)$ is $\begin{bmatrix} dim(Hull(C_1)) \\ l_1 \end{bmatrix}_2$ and the number of binary self-orthogonal codes V_2 such that $V_2 \subseteq Hull(C_2)$ is $\begin{bmatrix} dim(Hull(C_2)) \\ l_2 \end{bmatrix}_2$, by noting that each subcode of a binary self-orthogonal code is a binary self-orthogonal code. Therefore, by Theorem 4.20, for all possible l_1 and l_2 with $l_1 + l_2 = l$, the number of self-orthogonal codes $\beta(V)$ such that $\beta(V) \subseteq \beta(Hull(C))$ is

$$\sum_{l_1+l_2=l} \begin{bmatrix} dim(Hull(C_1)) \\ l_1 \end{bmatrix}_2 \cdot \begin{bmatrix} dim(Hull(C_2)) \\ l_2 \end{bmatrix}_2,$$

which gives the desired result. $\square$

Example 5. Let $C = CRT(C_1, C_2)$, where C_1 is generated by the set of vectors

$$\{(1,1,0,0), (0,0,1,1)\}$$

and C_2 is generated by the vector $(1,1,1,1)$. We have that C_1 is a binary linear code of length 4 and dimension 2 and C_2 is a binary linear code of length 4 and dimension 1. Since C_1 and C_2 are self-orthogonal codes; $Hull(C_i) = C_i$, for $i = 1, 2$.

The number of Euclidean self-orthogonal codes V of size 2^l such that $V \subset Hull(C)$, where $l = 1$ is equal to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_2 + \begin{bmatrix} 2 \\ 0 \end{bmatrix}_2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 = 3 + 1 = 4$. The generators of these codes are given as follows:

$$\begin{array}{ccc} \begin{pmatrix} v & v & v & v \end{pmatrix} & , & \begin{pmatrix} 1+v & 1+v & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 0 & 1+v & 1+v \end{pmatrix} & , & \begin{pmatrix} 1+v & 1+v & 1+v & 1+v \end{pmatrix}. \end{array}$$

The number of Euclidean self-orthogonal codes V of size 2^l such that $V \subset Hull(C)$, where $l=2$ is equal to $\begin{bmatrix} 2 \\ 2 \end{bmatrix}_2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_2 + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 + \begin{bmatrix} 2 \\ 0 \end{bmatrix}_2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}_2 = 1+3+0=4$.

The generators of these codes are given as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & v & v \end{pmatrix}, \begin{pmatrix} v & v & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1+v & 1+v & 0 & 0 \\ 0 & 0 & 1+v & 1+v \end{pmatrix}.$$

Note that $C = \{(0,0,0,0), (v,v,v,v), (1+v,1+v,0,0), (1,1,v,v), (0,0,1+v,1+v), (v,v,1,1), (1+v,1+v,1+v,1+v), (1,1,1,1)\}$ is a Euclidean self-orthogonal code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$, and so $Hull(C) = C$.

Lemma 4.23. *Let C be a linear code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n. Let $V \subseteq Hull(C)$ be a Euclidean self-orthogonal code of length n and size 2^l such that $CRT(V_1, V_2) = V$ where V_1 and V_2 are self-orthogonal binary codes. The number of linear codes C of length n and size 2^k such that $V \subseteq Hull(C)$ is equal to*

$$\sum_{k_1+k_2=k} \begin{bmatrix} n-2l_1 \\ k_1-l_1 \end{bmatrix}_2 \cdot \begin{bmatrix} n-2l_2 \\ k_2-l_2 \end{bmatrix}_2,$$

where C_1 and C_2 are binary linear codes such that $CRT(C_1, C_2) = C$ and k_i and l_i are the dimensions of the codes C_i and V_i with $l_i \leq k_i$, respectively, for $i = 1, 2$.

Proof. Let $CRT(V_1, V_2) = V$ be a Euclidean self-orthogonal code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n and size 2^l such that $V \subseteq Hull(C)$ where V_1 and V_2 are binary self-orthogonal codes of dimension l_1 and l_2, respectively. Let $CRT(C_1, C_2) = C$, where C_1 and C_2 are binary linear codes of dimension k_1 and k_2, respectively. Then by Theorem 4.19, we write $\beta(V) = V_1 \times V_2$ and $\beta(C) = C_1 \times C_2$, where $C_1 = \{\mathbf{x} \in \mathbb{F}_2^n | \mathbf{x} + v\mathbf{y} \in C, y \in \mathbb{F}_2^n\}$, $C_2 = \{\mathbf{x}+\mathbf{y} \in \mathbb{F}_2^n | \mathbf{x} + v\mathbf{y} \in C\}$ and $V_1 = \{\mathbf{x} \in \mathbb{F}_2^n | \mathbf{x} + v\mathbf{y} \in V, \mathbf{y} \in \mathbb{F}_2^n\}$ and $V_2 = \{\mathbf{x}+\mathbf{y} \in \mathbb{F}_2^n | \mathbf{x}+v\mathbf{y} \in V\}$. By the proof of Lemma 4.22, we have that $\beta(V) \subseteq \beta(Hull(C)) \subseteq \beta(C)$ and so $V_1 \times V_2 \subseteq Hull(C_1) \times Hull(C_2) \subseteq C_1 \times C_2$. Since all the linear codes are nonempty as having at least the zero vector, we have the following two relations:

$$V_1 \subseteq Hull(C_1) \subseteq C_1 \tag{6}$$

$$V_2 \subseteq Hull(C_2) \subseteq C_2. \tag{7}$$

By (6), we have that $V_1 \subseteq C_1 \cap C_1^\perp$ and so $V_1 \subseteq C_1 \subseteq V_1^\perp$. By [25], the number of codes C_1 such that $V_1 \subseteq C_1 \subseteq V_1^\perp$, is equal to $\begin{bmatrix} n-2l_1 \\ k_1-l_1 \end{bmatrix}_2$. Similarly by (7), the number of codes C_2 such that $V_2 \subseteq C_2 \subseteq V_2^\perp$, is equal to $\begin{bmatrix} n-2l_2 \\ k_2-l_2 \end{bmatrix}_2$. Therefore, by Theorem 4.20, for all possible k_1 and k_2 such that $k_1+k_2=k$, the number of codes C such that $V \subseteq Hull(C)$ is equal to

$$\sum_{k_1+k_2=k} \begin{bmatrix} n-2l_1 \\ k_1-l_1 \end{bmatrix}_2 \cdot \begin{bmatrix} n-2l_2 \\ k_2-l_2 \end{bmatrix}_2,$$

by noting that β is an isomorphism and C is a linear code which is uniquely determined by the binary linear codes C_1 and C_2, □

Example 6. Let $V = \{(0,0), (v,v)\}$. The code V is a Euclidean self-orthogonal code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$. Let $V = CRT(V_1, V_2)$, then V_1 is generated by the vector $(0,0)$ and V_2 is generated by the vector $(1,1)$. The dual of V_1 is the code $\mathbb{F}_2^2$. By the relation $V_1 \subseteq C_1 \subseteq V_1^{\perp}$, the number of binary linear codes C_1 of length 2 and dimension 1 is equal to $\begin{bmatrix} n-2l_1 \\ k_1-l_1 \end{bmatrix}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_2 = 3$. These binary codes are generated by the following matrices:

$$\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

The dual of V_2 is itself since it is a self-dual code. By the relation $V_2 \subseteq C_2 \subseteq V_2^{\perp}$, the number of binary linear codes C_2 of length 2 and dimension 1 is equal to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_2 = 1$. This binary code is the code $C_2 = V_2$.

Therefore, by Theorem 4.20, the number of linear codes of length n and size 2^2 such that $V \subseteq Hull(C)$ is equal to $3 \cdot 1 = 3$. These codes, D_1, D_2, D_3, are exactly the codes which are the CRT of the codes C_1 and C_2. They are given as follows: $D_1 = CRT(C_1, C_2) = \{(0,0),(v,1),(v,v),(0,1+v)\}$ where $C_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}$.

The code $D_2 = CRT(C_1, C_2) = \{(0,0),(1,v),(v,v),(1+v,0)\}$ where $C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}$.

The code $D_3 = CRT(C_1, C_2) = \{(0,0),(1,1),(v,v),(1+v,1+v)\}$ where $C_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} = C_2$.

For the codes D_1, D_2, D_3, we have that $V \subseteq Hull(D_i)$, where V is generated by the vector (v,v), and $Hull(D_i)$ is generated by (v,v), for $i = 1,2$ and generated by $(1,1)$, for $i = 3$. Note that the only possible dimension for C_2 is 1, so the only possible dimension for C_1 for the considered case is 1, even though C_1 may have dimensions $0, 1$ or 2.

Theorem 4.24. *Let $\sigma_{n,l}$ be the number of Euclidean self-orthogonal codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n and size 2^l. Then we have,*

$$\sigma_{n,l} = \sum_{k=k_1+k_2} \sum_{l=l_1+l_2} \frac{\left(\sum_{i=l_1}^{k_1} \begin{bmatrix} i \\ l_1 \end{bmatrix}_2 \cdot \mathfrak{N}_{n,k_1,i}\right)\left(\sum_{j=l_2}^{k_2} \begin{bmatrix} j \\ l_2 \end{bmatrix}_2 \cdot \mathfrak{N}_{n,k_2,j}\right)}{\begin{bmatrix} n-2l_1 \\ k_1-l_1 \end{bmatrix}_2 \cdot \begin{bmatrix} n-2l_2 \\ k_2-l_2 \end{bmatrix}_2}, \quad (8)$$

where $\mathfrak{N}_{n,k_1,i}$ and $\mathfrak{N}_{n,k_2,j}$ denote the number of binary linear codes with the hull dimensions i and j, for all $l_1 \leq i \leq k_1$, $l_2 \leq j \leq k_2$, $l_1 + l_2 = l$ and i, j, k, k_1, k_2, l are positive integers.

Proof. Let C be a linear code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n size 2^l. By Lemma 4.22, $Hull(C)$ contains $\sum_{l_1+l_2=l} \begin{bmatrix} dim(Hull(C_1)) \\ l_1 \end{bmatrix} \cdot \begin{bmatrix} dim(Hull(C_2)) \\ l_2 \end{bmatrix}$ Euclidean self-orthogonal codes of size 2^l, where $CRT(C_1, C_2) = C$ and C_1 and C_2 are binary linear codes given in Theorem 4.19 and $l = l_1 + l_2$. By Lemma 4.23, any Euclidean self-orthogonal code over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of size 2^l, for $l = l_1 + l_2$, is contained in the hull of $\sum_{k_1+k_2=k} \begin{bmatrix} n-2l_1 \\ k_1-l_1 \end{bmatrix}_2 \cdot \begin{bmatrix} n-2l_2 \\ k_2-l_2 \end{bmatrix}_2$ different linear codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of size 2^k. Therefore, by Theorem 4.21, the number of Euclidean self-orthogonal codes, $\sigma_{n,l}$, over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of size 2^l is equal to

$$\sigma_{n,l} = \frac{\left(\sum\limits_{\substack{C_1\subseteq \mathbb{F}_2^n\\ dim(C_1)=k_1}} \begin{bmatrix} dim(Hull(C_1)) \\ l_1 \end{bmatrix}_2\right)\left(\sum\limits_{\substack{C_2\subseteq \mathbb{F}_2^n\\ dim(C_2)=k_2}} \begin{bmatrix} dim(Hull(C_2)) \\ l_2 \end{bmatrix}_2\right)}{\sum_{k_1+k_2=k} \begin{bmatrix} n-2l_1 \\ k_1-l_1 \end{bmatrix}_2 \cdot \begin{bmatrix} n-2l_2 \\ k_2-l_2 \end{bmatrix}_2}, \tag{9}$$

where $k=k_1+k_2$, k_i is the dimension of binary linear codes C_i, for $i=1,2$, and $l=l_1+l_2$, l_i is the dimension of the self-orthogonal codes V_i, respectively, such that $V_i\subset Hull(C_i)$, for $i=1,2$, by noting that the number of Euclidean self-orthogonal codes, $\sigma_{n,l}$, over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ is equal to the product of binary self-orthogonal codes V_i of dimension l_i, for $i=1,2$, by Theorem 4.20 and Theorem 4.21.

We rewrite the numerator of (9), by taking $\mathfrak{N}_{n,k_1,i}$ and $\mathfrak{N}_{n,k_2,j}$ as the number of binary linear codes of dimension k_i with a hull of dimension l_i for $i=1,2$, respectively. Then considering for all $l_1\le i\le k_1$, $l_2\le j\le k_2$ and $l_1+l_2=l$, we get the result. □

Note that the number given by (8), is the multiplication of the number of binary self orthogonal codes over of length n and dimension l_1 and the number of binary self orthogonal codes over of length n and dimension l_2.

The next theorem is our desired result. Namely, it gives the number of codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ that have a hull of a given size.

Theorem 4.25. *Let $k\le n/2$ and $l\le k$. The number of linear codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n and size 2^k where the size of the hull is 2^l is equal to*

$$\begin{aligned} &\sum_{k=k_1+k_2}\sum_{l=l_1+l_2} \mathfrak{N}_{n,k_1,l_1}\cdot \mathfrak{N}_{n,k_2,l_2} \\ &= \sum_{k=k_1+k_2}\sum_{l=l_1+l_2}\left(\sum_{i=l_1}^{k_1}\begin{bmatrix} n-2i \\ k_1-i\end{bmatrix}\begin{bmatrix} i \\ l_1\end{bmatrix}(-1)^{i-l_1}2^{\binom{i-l_1}{2}}\sigma_{n,i}\right) \\ &\cdot\left(\sum_{j=l_2}^{k_2}\begin{bmatrix} n-2j \\ k_2-j\end{bmatrix}\begin{bmatrix} j \\ l_2\end{bmatrix}(-1)^{j-l_2}2^{\binom{j-l_2}{2}}\sigma_{n,j}\right)\end{aligned}$$

where $\sigma_{n,i}$ is the number of binary self-orthogonal codes of length n and size 2^i.

Proof. It follows from Theorem 4.6 and Theorem 4.24. □

Table 1 illustrates the theorem for some n,k and l. Note that we are only interested in even lengths. The number for $(n,k,l)=(n,k,0)$ gives the number of codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ when the hull is trivial. These codes are in fact LCD codes. These numbers for $l=0$, which can be seen on Table 1, give the following sequence in [29] by reference number A000302: $4,16,64,256,1024,\ldots$. Next, in particular, we give two examples in detail.

Example 7. Let $n=2$, $k=1$ and $l=1$. Then, we have that $(k_1,k_2)\in\{(1,0),(0,1)\}$ and $(l_1,l_2)\in\{(1,0),(0,1)\}$, with $l_i\le k_i$, for $i=1,2$. Then the number of linear codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length 2 and size 2^1 with a hull of size 2^1 is 2, by Theorem 4.25. These codes are generated by the following matrices:

$$\begin{pmatrix} v & v\end{pmatrix},\ \begin{pmatrix} 1+v & 1+v\end{pmatrix}.$$

Table 1. The number of codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length n and size 2^k, with a hull of size 2^l

n	k	l	Number	n	k	l	Number	n	k	l	Number
2	0	0	1	**8**	0	0	1	**10**	0	0	1
	1	0	4		1	0	256		1	0	1024
	1	1	2		1	1	254		1	1	1022
4	0	0	1		2	0	27264		2	0	436736
	1	0	16		2	1	40576		2	1	653824
	1	1	14		2	2	18775		2	2	304471
	2	0	104		3	0	1478656		3	0	94961664
	2	1	136		3	1	2499296		3	1	161622912
	2	2	55		3	2	1382976		3	2	90282240
6	0	0	1		3	3	338832		3	3	22346160
	1	0	64		4	0	40786432		4	0	$10^{10}\cdot 10520$
	1	1	62		4	1	65877504		4	1	$10^{10}\cdot 17134$
	2	0	1696		4	2	44123352		4	2	$10^{10}\cdot 11603$
	2	1	2464		4	3	13590432		4	3	$10^{9}\cdot 36314$
	2	2	1111		4	4	2104929		4	4	569194425
	3	0	22784						5	0	$10^{11}\cdot 51070$
	3	1	37432						5	1	$10^{11}\cdot 89580$
	3	2	199206						5	2	$10^{11}\cdot 63325$
	3	3	4680						5	3	$10^{11}\cdot 23516$
									5	4	$10^{10}\cdot 43198$
									5	5	$10^{9}\cdot 42609$

We note that $\sigma_{2,1}=1$, by Theorem 4.3.

Example 8. Let $n=4$, $k=2$ and $l=1$. Then, we have that $(k_1,k_2)\in\{(2,0),(1,1),(0,2)\}$ and $(l_1,l_2)\in\{(1,0),(0,1)\}$. We have following 3 cases:

1. $(k_1,k_2)=(2,0)$ and $(l_1,l_2)=(1,0)$.

$$\left(\sum_{i=1}^{2}\begin{bmatrix}4-2i\\2-i\end{bmatrix}\begin{bmatrix}i\\1\end{bmatrix}(-1)^{i-1}2^{\binom{i-1}{2}}\sigma_{4,i}\right)\left(\sum_{j=0}^{0}\begin{bmatrix}4-2j\\0-j\end{bmatrix}\begin{bmatrix}j\\0\end{bmatrix}(-1)^{j-0}2^{\binom{j-0}{2}}\sigma_{4,j}\right)=12.$$

2. $(k_1,k_2)=(1,1)$ and $(l_1,l_2)=(1,0)$, $(l_1,l_2)=(0,1)$.

$$\left(\sum_{i=1}^{1}\begin{bmatrix}4-2i\\1-i\end{bmatrix}\begin{bmatrix}i\\1\end{bmatrix}(-1)^{i-1}2^{\binom{i-1}{2}}\sigma_{4,i}\right)\left(\sum_{j=0}^{1}\begin{bmatrix}4-2j\\1-j\end{bmatrix}\begin{bmatrix}j\\0\end{bmatrix}(-1)^{j-0}2^{\binom{j-0}{2}}\sigma_{4,j}\right)+$$
$$\left(\sum_{i=0}^{1}\begin{bmatrix}4-2i\\1-i\end{bmatrix}\begin{bmatrix}i\\0\end{bmatrix}(-1)^{i-0}2^{\binom{i-0}{2}}\sigma_{4,i}\right)\left(\sum_{j=1}^{1}\begin{bmatrix}4-2j\\1-j\end{bmatrix}\begin{bmatrix}j\\1\end{bmatrix}(-1)^{j-1}2^{\binom{j-1}{2}}\sigma_{4,j}\right)$$
$$(7)(8)+(8)(7)=\quad 112.$$

3. $(k_1,k_2)=(0,2)$ and $(l_1,l_2)=(0,1)$.

$$\left(\sum_{i=0}^{0}\begin{bmatrix}4-2i\\0-i\end{bmatrix}\begin{bmatrix}i\\0\end{bmatrix}(-1)^{i-0}2^{\binom{i-0}{2}}\sigma_{4,i}\right)\left(\sum_{j=1}^{2}\begin{bmatrix}4-2j\\2-j\end{bmatrix}\begin{bmatrix}j\\1\end{bmatrix}(-1)^{j-1}2^{\binom{j-1}{1}}\sigma_{4,j}\right)=12.$$

Therefore, the number of linear codes over $\mathbb{F}_2[v]/\langle v^2+v\rangle$ of length 4 and size 2^2 with a hull of size 2^1 is $12+112+12=136$, by Theorem 4.25. We note that $\sigma_{4,0}=1$, $\sigma_{4,1}=7$ and $\sigma_{4,2}=3$ by Theorem 4.3.

4.3. **Counting additive codes over $\mathbb{F}_2[u]/\langle u^2\rangle$.** In this section, we shall count additive codes over $\mathbb{F}_2[u]/\langle u^2\rangle$.

Consider the following duality on the additive group of $\mathbb{F}_2[u]/\langle u^2\rangle$:

	0	1	u	$1+u$
0	1	1	1	1
1	1	-1	1	-1
u	1	-1	1	-1
$1+u$	1	1	-1	-1

We note that this duality is often called the Euclidean duality.

Lemma 4.26. *Let* $\mathbf{v}, \mathbf{w} \in \mathbb{F}_2[u]/\langle u^2\rangle^n$. *Then* $[\mathbf{v}, \mathbf{w}]_M = 1$ *if and only if* $[\psi(\mathbf{v}), \beta(\mathbf{w})] = 0$.

Proof. The result follows by examining the table of inner-products of the image of the elements of $\mathbb{F}_2[u]/\langle u^2\rangle$, under ψ.

	00	01	11	10
00	0	0	0	0
01	0	1	0	1
11	0	1	0	1
10	0	0	1	1

□

Theorem 4.27. *Let C be an additive code over $\mathbb{F}_2[u]/\langle u^2\rangle$. Then* $Hull(\psi(C)) = \psi(Hull_{M_E}(C))$.

Proof. The proof is the same as Theorem 3.2 using Lemma 4.26. □

This leads to the following important theorem for additive codes over $\mathbb{F}_2[u]/\langle u^2\rangle$.

Theorem 4.28. *The number of additive codes over $\mathbb{F}_2[u]/\langle u^2\rangle$ of length n where $Hull_{M_E}(C)$ has size 2^k is equal to the number of binary linear codes of length $2n$ with hulls of dimension k.*

Proof. Every additive code over $\mathbb{F}_2[u]/\langle u^2\rangle$ of length n is of the form $\psi^{-1}(C)$ for some linear binary code of length n. Then by Theorem 4.27 we have the result. □

Theorem 4.29. *The number of additive codes over $\mathbb{F}_2[u]/\langle u^2\rangle$, $u^2 = 0$, of length n and size 2^k whose hull size is 2^l, $l \le k$, is*

$$\sum_{i=l}^{k} \begin{bmatrix} 2n-2i \\ k-i \end{bmatrix} \begin{bmatrix} i \\ l \end{bmatrix} (-1)^{i-l} 2^{\binom{i-l}{2}} \sigma_{2n,i}.$$

Proof. It follows from Theorem 4.6 and Theorem 4.28. □

Example 9. Let $n = 2$, $k = 2$ and $l = 1$. Then by Theorem 4.3, $\sigma_{4,1} = 7$ and $\sigma_{4,2} = 3$. The number of additive codes over $\mathbb{F}_2[u]/\langle u^2\rangle$ of length 2 and size 2^2 whose hull has size 2^1 is 12 by Theorem 4.29. These are the additive codes given by the following generating matrices:

$$\begin{pmatrix} 1+u & 0 \\ 1 & 1+u \end{pmatrix}, \begin{pmatrix} 1+u & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1+u & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1+u & 1+u \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1+u & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} 0 & 1+u \\ u & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1+u \\ 1+u & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1+u \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1+u & 1+u \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1+u \end{pmatrix}.$$

Corollary 4.30. *Let R be one of the rings $\mathbb{F}_4$, $\mathbb{F}_2[v]/\langle v^2+v\rangle$ or $\mathbb{F}_2[u]/\langle u^2\rangle$. Then the number of additive codes over the ring R of length n and size 2^k whose hull size is equal for each ring R.*

Proof. It follows from Theorem 4.9, Theorem 4.10, Theorem 4.16 and Theorem 4.29. □

4.4. **Counting linear codes over $\mathbb{Z}_4$ and $\mathbb{F}_2[u]/\langle u^2\rangle$.** In this section, we shall count free linear codes over the two non-trivial chain rings of order r, namely $\mathbb{F}_2[u]/\langle u^2\rangle$ and $\mathbb{Z}_4$. (The field $\mathbb{F}_4$ is also a chain ring but trivially so, as it has no non-trivial ideals.) Since they are both chain rings we can handle them in a similar manner.

Let $R=\mathbb{Z}_4$ or $R=\mathbb{F}_2[u]/\langle u^2\rangle$. Then any element in R is denoted by $a+xb$ where $x=2$ for $\mathbb{Z}_4$ and $x=u$ for $\mathbb{F}_2[u]/\langle u^2\rangle$. The main difference between the rings is that the characteristic of $\mathbb{F}_2[u]/\langle u^2\rangle$ is 2, and the characteristic of $\mathbb{Z}_4$ is 4.

Throughout this section, unless otherwise stated, $R=\mathbb{Z}_4$ or $R=\mathbb{F}_2[u]/\langle u^2\rangle$.

A vector over R is called free if it has at least one component 1 or 3 for $R=\mathbb{Z}_4$ and 1 or $1+u$ for $R=\mathbb{F}_2[u]/\langle u^2\rangle$ and otherwise it is called a non-free vector. In other words, a vector is free if it generates a code of size 4. A code is free if every non-free vector is a multiple of a free vector. It follows that if a code is free over the ring R then the code is isomorphic to R^k for some k. The next theorem gives the number of linear codes over R of type (l_0,l_1) of a linear code over R of type (k_0,k_1).

Theorem 4.31. *Let C be a linear code over R of type (k_0,k_1). The number of subcodes of C over R of type (l_0,l_1) is equal to*

$$\frac{2^{k_0l_0+k_1l_0}\prod_{i=0}^{l_0-1}(2^{k_0}-2^i)\prod_{j=0}^{l_1-1}(2^{k_0+k_1}-2^{l_0+j})}{2^{l_0^2+2l_0l_1}\prod_{i=0}^{l_0-1}(2^{l_0}-2^i)\prod_{j=0}^{l_1-1}(2^{l_1}-2^j)}. \tag{10}$$

Proof. We apply Theorem 4.2, by considering the code C of type (k_0,k_1) as the whole space itself from which we are choosing vectors to construct codes of type (l_0,l_1). □

Denote the number given by (10) by $\begin{bmatrix} k_0,k_1 \\ l_0,l_1 \end{bmatrix}_R$. Note that for $l_0<0$ or $l_1<0$, we have that $\begin{bmatrix} k_0,k_1 \\ l_0,l_1 \end{bmatrix}_R=0$ and for $l_0=l_1=0$, we have that $\begin{bmatrix} k_0,k_1 \\ 0,0 \end{bmatrix}_R=1$.

If $k_1=0$ or $l_1=0$, then we denote the number by $\begin{bmatrix} k_0 \\ l_0,l_1 \end{bmatrix}_R$ instead of $\begin{bmatrix} k_0,0 \\ l_0,l_1 \end{bmatrix}_R$ and $\begin{bmatrix} k_0,k_1 \\ l_0 \end{bmatrix}_R$ instead of $\begin{bmatrix} k_0,k_1 \\ l_0,0 \end{bmatrix}_R$, respectively.

Note also that for a linear subcode D of type (l_0,l_1) of a linear code C over R of type (k_0,k_1), we have that $2l_0+l_1\le 2k_0+k_1$ and $l_0\le k_0$. The first statement follows from the fact that D is a linear subcode of C. The latter is because of that any linear subcode with l_0 free vectors is contained in a code with at least l_0 free vectors. However, the inequality $l_1\le k_1$ need not be satisfied. Consider the code C over $\mathbb{Z}_4$ generated by $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ which has type $(1,1)$. The subcode D of C generated by $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has type $(0,2)$ and $2 \not\leq 1$.

Let V be a self-orthogonal code over R of type (l_0, l_1) such that $V \subseteq Hull(C)$. Then $V^\perp$ has type $(n-l_0-l_1, l_1)$. Since $V \subseteq Hull(C)$, we have that $V \subseteq C \subseteq V^\perp$. Then the code C has type (k_0, k_1), where $k_0 = l_0 + s$ and $k_1 = l_1 + t$, for some integer s and t. However, this does not mean that we are necessarily adjoining $s+t$ generators to V to obtain C. For example, over $\mathbb{Z}_4$, let $V = \{(0,0),(2,2)\}$, which is a non-free linear code of type $(0,1)$. Then $V^\perp$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

and has type $(1,1)$. That is, we have replaced the non-free vector $(2,2)$ with the free vector $(1,1)$ as a generator. This ensures that $(2,2)$ is still in the code but the code $\langle(1,1)\rangle$ has type $(1,0)$ and the code $\langle(2,2)\rangle$ has type $(0,1)$. Then we adjoin the non-free vector $(0,2)$ to get a code of type $(1,1)$.

Specifically, one possible types for the code C is $(1,0)$ and it is obtained by adjoining one vector into the free generators and removing one vector from the non-free generators, that is $s = 1$ and $t = -1$. This can be done because $(1,1) \in V^\perp$ and $(2,2) \in V$ and so C is generated by one of the following vectors $(1,1),(1,3),(3,1)$. This example shows one of the cases when V is any self- orthogonal code over $\mathbb{Z}_4$ of type (l_0,l_1). A similar case is satisfied for codes over $\mathbb{F}_2[u]/\langle u^2\rangle$, by simply replacing the 2 with a u. This type of situation substantially complicates the counting process. Therefore, in this work, we focus on free Euclidean self-orthogonal codes V over R and construct linear codes C and so leave the general case for future study.

Lemma 4.32. *Let C be a linear code over R, where $R = \mathbb{Z}_4$ or $R = \mathbb{F}_2[u]/\langle u^2\rangle$. Let $Hull(C)$ be its hull of type (h_0, h_1). The number of free Euclidean self-orthogonal codes over R with l_0 generators such that $V \subseteq Hull(C)$ is equal to*

$$\begin{bmatrix} h_0, h_1 \\ l_0 \end{bmatrix}_R .$$

Proof. Since $Hull(C)$ is a Euclidean self-orthogonal code over R, each of its subcodes is also Euclidean self-orthogonal over R. Hence by Theorem 4.31, counting free subcodes of the hull gives the desired number. □

Lemma 4.33. *Let V be a free Euclidean self-orthogonal code over R of length n with l_0 generators, where $R = \mathbb{Z}_4$ or $R = \mathbb{F}_2[u]/\langle u^2\rangle$. The number of linear codes C over R of length n and type (k_0,k_1) such that $V \subseteq Hull(C)$ is*

$$\begin{bmatrix} n-2l_0 \\ k_0-l_0, k_1 \end{bmatrix}_{2,2} .$$

Proof. Let V be a free Euclidean self-orthogonal code over R of length n with l_0 generators. Then V has type $(l_0, 0)$ and $V^\perp$ is a free linear code and has type $(n-l_0, 0)$. By $V \subseteq Hull(C)$, we have that $V \subseteq C \subseteq V^\perp$. Then C is a linear code over R of length n and type (k_0,k_1). It is clear that $l_0 \leq k_0 \leq n-l_0$ and $k_1 \leq n-k_0-l_0$. Therefore, to construct C, we are choosing $k_0 - l_0$ free and k_1 non-free generators among $n - 2l_0$ free generators of $V^\perp$. Therefore, we apply Theorem 4.31 and get the number $\begin{bmatrix} n-2l_0 \\ k_0-l_0, k_1 \end{bmatrix}_R = \begin{bmatrix} n-2l_0 \\ k_0-l_0, k_1 \end{bmatrix}_{2,2}$. □

TABLE 2. The number of codes C when V has length $n = 4$ and type $= (1, 0)$

(k_0, k_1)	N	Generators
(1,0)	1	V
(1,1)	3	$\langle V, 2200\rangle$, $\langle V, 2020\rangle$, $\langle V, 2002\rangle$.
(1,2)	1	$\langle V, 2200, 2020\rangle$
(2,0)	6	$\langle V, 1300\rangle$,$\langle V, 1030\rangle$, $\langle V, 1003\rangle$, $\langle V, 0130\rangle$, $\langle V, 0103\rangle$, $\langle V, 0013\rangle$
(2,1)	3	$\langle V, 1300, 2020\rangle$, $\langle V, 1030, 2200\rangle$, $\langle V, 1003, 2020\rangle$
(3,0)	1	$\langle V, 1300, 1030\rangle = V^\perp$

Note that if V is a free Euclidean self-orthogonal code over $\mathbb{Z}_4$ then its length is at least 4, so we consider linear codes C over $\mathbb{Z}_4$ of length at least 4. This is not the case when we work codes over $\mathbb{F}_2[u]/\langle u^2\rangle$. Any free Euclidean self-orthogonal code over $\mathbb{F}_2[u]/\langle u^2\rangle$ has at least length 2. For instance, the code generated by $\langle(1, 1)\rangle$ is a free self-orthogonal code of length 1.

Example 10. Let $V = \{(0, 0, 0, 0), (1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3)\}$ which is a linear code over $\mathbb{Z}_4$ of type $(1, 0)$. The code $V^\perp$ has type $(3, 0)$ and it is generated by

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}.$$

Then C has type (k_0, k_1) such that $V \subseteq Hull(C)$, where $k_0 + k_1 \leq 3$.

Possible values for (k_0, k_1) are $(1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (3, 0)$. Then, by Lemma 4.33, the number of codes C for each type are given in Table 2.

Theorem 4.34. *Let σ_{l_0} be the number of free Euclidean self-orthogonal codes over R of type $(l_0, 0)$, where $R = \mathbb{Z}_4$ or $R = \mathbb{F}_2[u]/\langle u^2\rangle$. Let C be a linear code over R of length n and type (k_0, k_1). Let $N^{k_0,k_1}_{i_0,i_1}$ denote the number of linear codes over R of length n and type (k_0, k_1) whose hull has type (i_0, i_1). Then for all codes C of type (k_0, k_1), we have the following*

$$\sigma_{l_0} = \frac{\sum\limits_{\substack{i_0=l_0,\ldots,k_0 \\ i_1=0,\ldots,k_0+k_1-i_0}} \begin{bmatrix} i_0, i_1 \\ l_0 \end{bmatrix}_R N^{k_0,k_1}_{i_0,i_1}}{\begin{bmatrix} n-2l_0 \\ k_0-l_0, k_1 \end{bmatrix}_{2,2}} \tag{11}$$

Proof. Let C be a linear code over R of length n and type (k_0, k_1) with the hull of type (h_0, h_1). Then, by Lemma 4.32, $Hull(C)$ contains $\begin{bmatrix} k_0, k_1 \\ l_0 \end{bmatrix}_R$ free Euclidean self-orthogonal codes of type $(l_0, 0)$. Any free Euclidean self-orthogonal code is contained in the hull of $\begin{bmatrix} n-2l_0 \\ k_0-l_0, k_1 \end{bmatrix}_{2,2}$ distinct linear codes C of type (k_0, k_1), by Lemma 4.33. Then the number of distinct free Euclidean self-orthogonal codes σ_{l_0} is equal to

$$\sigma_{l_0} = \frac{\displaystyle\sum_{\substack{C\subseteq R^n\\ Type(C)=(k_0,k_1)\\ Type(Hull(C))=(h_0,h_1)}} \begin{bmatrix} h_0, h_1 \\ l_0 \end{bmatrix}_R}{\begin{bmatrix} n-2l_0 \\ k_0-l_0, k_1 \end{bmatrix}_{2,2}},$$

by considering all the codes over R of type (k_0, k_1). Then if $N^{k_0,k_1}_{i_0,i_1}$ denote the number of linear codes over R of length n and type (k_0, k_1) whose hull has type (i_0, i_1), we have the result. □

Next, we give an example to illustrate the theorem.

Example 11. Let $(k_0, k_1) = (2, 0)$ and $l_0 = 1$. The number of free Euclidean self-orthogonal codes over $\mathbb{Z}_4$ of length 4 is $\sigma_1 = 8$. These codes are generated by the following vectors:

$(1, 1, 1, 1), (1, 1, 1, 3), (1, 1, 3, 1), (1, 3, 1, 1), (1, 3, 3, 3)(1, 1, 3, 3), (1, 3, 1, 3), (1, 3, 3, 1)$.

We need the following calculations to get the number from the formula given by (11).

$$\begin{bmatrix} 2 \\ 1,0 \end{bmatrix}_{\mathbb{Z}_4} = \begin{bmatrix} 2 \\ 1,0 \end{bmatrix}_{2,2} = 6, \quad \begin{bmatrix} 1,1 \\ 1 \end{bmatrix}_{\mathbb{Z}_4} = 2.$$

Then by (11), we get

$$\begin{aligned}
8 &= \frac{\displaystyle\sum_{\substack{i_0=1,2\\ i_1=0,1}} \begin{bmatrix} i_0, i_1 \\ l_0 \end{bmatrix}_{\mathbb{Z}_4} N^{2,0}_{i_0,i_1}}{\begin{bmatrix} 2 \\ 1,0 \end{bmatrix}_{2,2}} \\
\Rightarrow 48 &= \begin{bmatrix} 1,0 \\ 1 \end{bmatrix}_{\mathbb{Z}_4} N^{2,0}_{1,0} + \begin{bmatrix} 1,1 \\ 1 \end{bmatrix}_{\mathbb{Z}_4} N^{2,0}_{1,1} + \begin{bmatrix} 2,0 \\ 1 \end{bmatrix}_{\mathbb{Z}_4} N^{2,0}_{2,0} \\
48 &= \begin{bmatrix} 1,1 \\ 1 \end{bmatrix}_{\mathbb{Z}_4} N^{2,0}_{1,1} \\
\Rightarrow 24 &= N^{2,0}_{1,1},
\end{aligned}$$

where $N^{2,0}_{1,0} = N^{2,0}_{2,0} = 0$.

$C^{\perp}$ has type $(2, 0)$ Therefore the number of linear codes over $\mathbb{Z}_4$ of length n and type $(2, 0)$ with a hull of type $(1, 1)$ is 24. These codes are given by the following generating matrices:

$$\begin{gathered}
\langle(1, 1, 1, 1), \mathbf{x}_1\rangle, \langle(1, 1, 1, 3), \mathbf{x}_2\rangle, \langle(1, 1, 3, 1), \mathbf{x}_3\rangle, \\
\langle(1, 3, 1, 1), \mathbf{x}_4\rangle, \langle(1, 1, 3, 3), \mathbf{x}_5\rangle, \langle(1, 3, 1, 3), \mathbf{x}_6\rangle.
\end{gathered}$$

where $\mathbf{x}_i$ is one of the vectors given on the corresponding right hand side, for $i = \{1, 2, 3, 4, 5, 6\}$:

$$\begin{aligned}
\mathbf{x}_1 &= (1, 3, 0, 0), (1, 0, 3, 0), (1, 0, 0, 3), (0, 1, 3, 0), (0, 1, 0, 3), (0, 0, 1, 3), \\
\mathbf{x}_2 &= (1, 3, 0, 0), (1, 0, 3, 0), (0, 1, 3, 0), (1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1), \\
\mathbf{x}_3 &= (1, 0, 0, 3), (0, 1, 0, 3), (1, 0, 1, 0), (0, 1, 1, 0), \\
\mathbf{x}_4 &= (0, 0, 1, 3), (1, 1, 0, 0),
\end{aligned}$$

$$\begin{aligned}\mathbf{x}_5 &= (1,0,1,0),(0,1,1,0),(1,0,0,1),(0,1,0,1),\\ \mathbf{x}_6 &= (1,1,0,0),(0,0,1,1).\end{aligned}$$

Note that for the above example, $N_{1,0}^{2,0} = N_{2,0}^{2,0} = 0$. Because C contains a free Euclidean self-orthogonal code V over $\mathbb{Z}_4$ of length 4. Since V is a free Euclidean self-orthogonal code, we have that the Hamming weight of the only free generator $\mathbf{v}$ is 4. The vector $\mathbf{v}$ is also a generator of C, where C is generated by $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\mathbf{u}$ is also a free vector. Then since $V \subset Hull(C)$, we have that $\mathbf{v}$ is also a generator of $C^{\perp}$. Therefore, $[\mathbf{u}, \mathbf{v}] = 0$ and so $w_H(\mathbf{u})$ is 2, 3 or 4. However, $w_H(\mathbf{u}) \neq 4$ since the type of C is $(2,0)$. If $w_H(\mathbf{u}) = 2$, then $[\mathbf{u}, \mathbf{u}] \neq 0$ since each of the nonzero components of $\mathbf{u}$ is 1 or 3 and if $w_H(\mathbf{u}) = 3$, then each of the nonzero components of $\mathbf{u}$ is 1, 2, or 3. Therefore, for both of the cases, $[\mathbf{u}, \mathbf{u}] \neq 0$ and so $\mathbf{u}$ is not contained in $C^{\perp}$ but $2\mathbf{u} \in C^{\perp}$ so the only type for the hull of C is $(1,1)$.

5. **Conclusion.** In this paper, we have given foundational results on the hull of both additive and linear codes over the four rings of order 4. This is motivated by by the fact that codes with very small hulls and very large hulls are highly desired. That is LCD (ACD for additive) codes and self-dual codes are highly interesting codes with numerous theoretical and practical applications. We have given formulas for counting codes in this setting with hulls of a given size. This allows us to count the number of LCD, ACD, self-orthogonal, and self-dual codes. This is highly important as it is the major tool used when making an exhaustive search of codes with certain characteristics.

For further avenues of research, one can consider the general case for non-free codes over $\mathbb{F}_2[u]/\langle u^2 \rangle$ and $\mathbb{Z}_4$. Additionally, one can generalize the techniques described here to count the number of linear and additive codes with hulls of a given type in the rings that are generalizations of these four rings. Namely, fields of order 4, the family of ring R_k, the family of rings A_k, and the rings $\mathbb{Z}_k$ (see [5] for a complete description of these families of rings).

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